DUKE UNIV. LIBRARY
DURHAM, N. C.

VOLUME 25 APRIL, 1939 NUMBER 4

PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

OF THE

UNITED STATES OF AMERICA

EXECUTIVE COMMITTEE

EDWIN BIDWELL WILSON, *Managing Editor*

RAYMOND PEARL R. G. HARRISON

PUBLISHED MONTHLY

Publication Office: Mack Printing Company, Easton, Pa.

Editorial Office: Harvard School of Public Health, Boston 17

Home Office of the Academy: Washington, D. C.

Entered as second-class matter June 21, 1920, at the post-office at Easton, Pennsylvania, under the act of August 24, 1912. Accepted for mailing at a special rate of postage, provided for in section 1103, act of October 3, 1917. Authorized June 18, 1920.

Price per annum, $5.00

OFFICERS OF THE ACADEMY

PRESIDENT ..... F. R. LILLIE
VICE PRESIDENT ..... A. L. DAY
FOREIGN SECRETARY ..... L. J. HENDERSON
HOME SECRETARY ..... FRED. E. WRIGHT
TREASURER ..... ARTHUR KEITH

EDITORIAL BOARD

The HOME SECRETARY and the FOREIGN SECRETARY of the ACADEMY
The CHAIRMAN OF THE NATIONAL RESEARCH COUNCIL

V. BUSH	L. J. HENDERSON	R. A. MILLIKAN
R. E. COKER	F. B. JEWETT	H. R. MOODY
A. L. DAY	A. B. LAMB	W. J. V. OSTERHOUT
L. P. EISENHART	E. R. LONG	F. K. RICHTMYER
C. E. GUTHE	C. R. LONGWELL	F. H. SEARES
	W. R. MILES	

INFORMATION TO SUSCRIBERS

SUBSCRIPTIONS TO THE PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES at the rate of $5.00 per annum should be made payable to the National Academy of Sciences, and set to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C. Single numbers, $0.50.

PAST publications of the NATIONAL ACADEMY OF SCIENCES from 1863 to 1926 and of the NATIONAL RESEARCH COUNCIL from 1916 to 1925 are listed in the Index to the First Ten Volumes of these PROCEEDINGS. This Index appeared as Part 2 of Volume 13, No. 1, January, 1927, and is for sale separately at the price of $1.50. Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

PAST volumes of the PROCEEDINGS may be obtained at five dollars per volume unless the sale of the volume would break a complete set of Volumes I to X. Single numbers may be obtained for fifty cents except where the sale of such numbers would break up a complete volume. Only two hundred complete sets of the PROCEEDINGS are available for sale—Volume I to X, price $75.00 (including Index, see *supra*). Orders should be sent to the Home Secretary, National Academy of Sciences, Constitution Ave. & 21st Street, Washington, D. C.

The following publications are issued by the NATIONAL RESEARCH COUNCIL. Orders and inquiries should be addressed: Publication Office, National Research Council, Constitution Ave. & 21st Street, Washington, D. C.

(A) THE BULLETIN OF THE NATIONAL RESEARCH COUNCIL published at irregular intervals. Individual numbers priced variously. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

(B) THE REPRINT AND CIRCULAR SERIES OF THE NATIONAL RESEARCH COUNCIL, individual numbers variously priced. A discount of 20% from the list price will be given to all who subscribe for the series as issued, direct from the Publication Office.

PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 25 April 15, 1939 Number 4
Copyright 1939 by The National Academy of Sciences

THE OCCURRENCE OF PARTHENOGENETIC DIPLOIDS IN TETRAPLOID MAIZE[1]

By L. F. Randolph and Harold E. Fischer

Division of Cereal Crops and Diseases, Bureau of Plant Industry, U. S. Department of Agriculture, and Cornell University Agricultural Experiment Station

Communicated March 15, 1939

Chromosome doubling within species and varieties of plants has contributed extensively to the origin of stable new forms among both cultivated and wild species. But the reverse process involving a reduction in chromosome numbers comparable to the reduction which occurs in the origination of parthenogenetic haploids from diploids is unknown among autopolyploids, with the possible exception of a case reported by Hakansson in Oenothera.[2] Evidence that this reverse process may occur is available from the results of intercrosses between intraspecific tetraploid and diploid races in which the tetraploid was the seed parent. Belling and Blakeslee[3] obtained diploids as well as triploids in a cross of this kind in Datura, and Gairdner and Darlington[4] noted the occurrence of a diploid plant from a similar cross in *Campanula persicifolia*. In experiments with tetraploid maize extending over a period of years diploid individuals have been noted at infrequent intervals in the progenies of tetraploid plants, but the possibility was not excluded that these were due to admixtures of diploid seed at planting time or to contaminations of some other sort.

Last year in experiments with tetraploid maize designed primarily to determine their frequency of occurrence 23 parthenogenetic maternal diploids were discovered among 17,165 individuals and their identity as such was definitely established by genetic tests and by direct cytological examination. Crosses were made between tetraploid stocks in which genes for recessive endosperm and seedling characters were contributed by the seed parent and their dominant alleles were contributed by the pollen parent. Any parthenogenetic maternal individuals originating pseudogamously from such crosses would exhibit the recessive characters of the seed parent and wolud therefore be readily distinguishable from the hybrids which would exhibit the dominant characteristics contributed by the pollen

parent. An attempt was not made to plan these crosses to aid in the identification of paternal as well as maternal parthenogenetic diploids, by introducing genetically unlike recessive seedling genes in both parent stocks.

Data were obtained from four different crosses, in each of which the seed parent was a green plant (A_1 *b pl* r^g) with colorless aleurone (r^g) and white endosperm (y_1). In the first cross sugary endosperm (su_1) also was involved.[5] The pollen parent of the first two crosses had purple plant color (A_1 *B Pl*) colored aleurone (*R*) and starchy (Su_1) yellow (Y_1) endosperm. The pollen parent of the third cross was a dilute sun-red plant (*A b pl* r^r) with yellow endosperm (Y_1), and that of the fourth cross was a sun-red plant (*A B pl* r^r) with yellow endosperm (Y_1). The genotype of each of the crosses and the number of parthenogenetic maternal diploids and hybrid tetraploids obtained from each were as follows:

		MAT. 2N	4N
(1)	A_1 *b pl* r^g su_1 y_1 × A_1 *B Pl* R^g Su_1 Y_1	3	904
(2)	A_1 *b pl* r^g y_1 × A_1 *B Pl* R^g Y_1	2	530
(3)	A_1 *b pl* r^g y_1 × A_1 *b pl* r^r Y_1	17	15,615
(4)	A_1 *b pl* r^g y_1 × A_1 *B Pl* r^r Y_1	1	93
		23	17,142

The first, second and fourth crosses were from hand-pollinated ears; the third cross was produced in an isolated plot, the seed parent being detasseled before pollen was shed. The F_1 populations were grown exclusively from seeds with hybrid endosperm, i.e., seeds exhibiting the dominant endosperm characters of the pollen parent.

The 23 individuals classed as maternal diploids were with respect to the contrasting characters contributed to the cross by the parent stocks genotypically like the tetraploid seed parent; they lacked anthocyanin color as seedlings and mature plants, and when selfed or crossed with appropriate tester stocks, exhibited the recessive endosperm and aleurone characters of the seed parent. The reduced size of their stomata and pollen indicated that they were diploid. They produced abundant good pollen, were highly self-fertile, crossed readily with other diploids and like diploids were highly cross-sterile with tetraploids.

Among 20 of the 23 exceptional plants in which accurate chromosome counts were made there were 15 plants with 20 chromosomes, 1 with 21 chromosomes, 3 with 22 chromosomes and 1 with 24 chromosomes. Since in tetraploid maize aberrant individuals with one or a few chromosomes more or less than the typical number 40 occur not infrequently,[6] it was anticipated that numbers greater than 20 would appear in some of the maternal diploids. These aberrant extra-chromosome plants furnish a readily available source of trisomic stocks, which would be especially favorable for a study of the phenotypic characteristics of the different trisomics provided

they originate from uniform inbred tetraploid stocks. They also furnish direct evidence of the frequency with which normal 20-chromosome gametes and functional extra-chromosome gametes are produced by the ovules of tetraploid plants.

In general the parthenogenetic maize diploids resembled in appearance and breeding behavior the diploid stocks from which their maternal tetraploid parents were derived. But a critical study of these relationships has not yet been made.

It is anticipated that these maternal diploids will provide valuable material for a new approach to the study of the cytogenetic, physiological and morphological consequence of direct chromosome doubling. It has been suggested by various workers that certain of the characteristics which distinguish autotetraploids from their parental diploids may have resulted from genotypic changes induced by chromosome doubling and are not due merely to the presence of the doubled number of chromosomes. This problem and others like it can be attacked by comparing maternal diploids extracted from autotetraploid stocks with the tetraploid parent and with the original diploid stock from which the tetraploid originated.

The occurrence of fertile parthenogenetic diploids in populations of autotetraploid plants is highly significant from the evolutionary standpoint, since these diploids are capable of establishing new races with the reduced chromosome number. Admixture with the parental tetraploid does not occur to any appreciable extent and seed is produced abundantly by the diploid from its own pollen to the exclusion of that from the tetraploid parent, according to the results of mixed pollen experiments with maize.[6] Equal significance cannot be attached to the occurrence of parthenogenetic diploids among allotetraploids which are of hybrid origin, because they are sterile.[7]

It would be inappropriate to discuss here the various conditions under which new species and horticultural varieties of plants originate by chromosome doubling or by the reverse process of halving the number of chromosomes. However, as a result of the discovery of fertile parthenogenetic diploids in the progeny of autotetraploid maize, the conclusion is inescapable that the evolutionary trend among autopolyploids is not necessarily unidirectional, the forms with low numbers being always primitive and those with high numbers being always derived from them as has been generally believed. It is now apparent that the trend may be in either direction. Polyploids which have arisen by direct chromosome doubling may in turn give rise to new forms with lower chromosome numbers. Maternal diploids occurred in tetraploid maize with a frequency in the ratio of approximately 1:750. If parthenogenetic diploids occurred with a similar frequency among autotetraploids generally there would be ample opportunity for new diploid races to originate from them.

[1] These experiments were aided by a grant to the senior author by the Committee on Effects of Radiation upon Living Organisms of the National Research Council.

[2] Hakansson (*Hereditas* **5**, 93–96 (1924)) discovered a 14-chromosome plant in the progeny of *Oenothera gigantea*, which was thought to have originated from the union of 7-chromosome gametes. But it is possible that the plant was a parthenogenetic diploid.

[3] Belling, John, and Blakeslee, A. F., *Amer. Nat.*, **58**, 60–61(1924).

[4] Gairdner, A. E., and Darlington, C. D., *Genetica*, **13**, 113–150 (1931).

[5] For an explanation of the gene symbols and a description of the characters referred to in this paper see Emerson, R. A., Beadle, G. W., and Fraser, A. C., *Cornell Univ. Agr. Expt. Sta. Memoir*, 180 (1935).

[6] Randolph, L. F., *Journ. Agr. Res.*, **50**, 591–605 (1935).

[7] The "haploid" derivatives of Aegilotricum are typical examples; cf. Katayama, Y., *Jap. Journ. Bot.*, **7**, 349–380 (1935).

Appendix.—Since this article was submitted for publication the current number of *Hereditas* was received, in which there was a paper by Heribert Nilsson (*Hereditas*, **25**, 1–8 (1939)) describing diploid derivatives of *Oenothera gigantea*, an autotetraploid form of *Oe. Lamarckiana* (cf. footnote 2). These derived diploids definitely exhibited certain characteristics of *Oe. gigantea* as well as certain characteristics of *Oe. Lamarckiana* and for this reason were appropriately designated *Oe. diplo-gigantea*.

THE SYNTHESIS AND DESTRUCTION OF VITAMIN B_1 BY PHYCOMYCES

By James Bonner and Edwin R. Buchman

William G. Kerckhoff Laboratories of the Biological Sciences, and Gates and Crellin Laboratories of Chemistry, California Institute of Technology

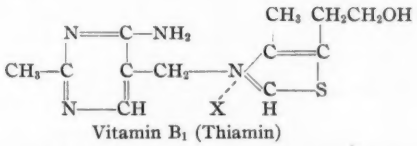
Communicated March 14, 1939

In an earlier communication[1] data relative to the *in vivo* synthesis of vitamin B_1 by the isolated pea root have been presented. For a fuller understanding of the physiological economy of the vitamin, some insight also into the modes of disappearance of the substance is indispensable. The accumulated evidence now permits us to present a picture both of *in vivo* synthesis of the vitamin by *Phycomyces Blakesleeanus* and of a mechanism by which the vitamin molecule is broken down by the latter organism.

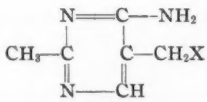
The experimental techniques were similar to those described in earlier communications.[1, 2] Stock cultures of *Phycomyces Blakesleeanus* and of *Phytophthora cinnamomi* were maintained on malt agar. The experimental cultures were, in all cases, made up with 10 cc. of medium ($MgSO_4.7H_2O$... 0.5 gm.; KH_2PO_4...1.5 gm.; asparagin...4.0 gm.; dextrose...100 gm.; distilled water...1 liter) to which was added the desired amount of vita-

min B_1 or related substances.[3] The experimental media were then autoclaved for 15 minutes at 15 pounds pressure, and inoculated, in the case of *Phycomyces*, with equal volumes of a sterile spore suspension, or, in the case of *Phytophthora*, with small pieces of mycelium. All cultures unless otherwise noted were allowed to remain 10 days at 25°C., the mycelium then filtered off, dried and weighed. All experiments were carried out in 3–5 fold replicate.

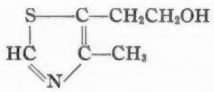
The structure of vitamin B_1 is shown below; the pyrimidine and thiazole portions of the molecule are also illustrated.



Vitamin B_1 (Thiamin)



Vitamin pyrimidine



Vitamin thiazole

It has been demonstrated repeatedly by various investigators[4] that *Phycomyces* is able to utilize not only the vitamin itself as a growth factor, but also equally well an equimolecular mixture of the pyrimidine (X = NH_2, Br, OC_2H_5) and thiazole fragments of the molecule. The *Phycomyces* assay determines then not only vitamin B_1 but also any uncombined pyrimidine plus thiazole which may be present in equimolecular amount in the sample under investigation.[5] *Phytophthora*[6] on the other hand utilizes vitamin B_1 as a growth factor but does not respond to a mixture of the two vitamin components. The *Phytophthora* assay, then, may be used for the determination of vitamin B_1 to the exclusion of any uncombined pyrimidine plus thiazole.

In the interpretation of experiments with *Phycomyces*, it has been assumed, but never demonstrated experimentally, that this organism synthesizes the vitamin from a mixture of vitamin pyrimidine and vitamin thiazole. That this is actually the case is shown by the following experiment. *Phycomyces* cultures supplied with an equimolecular mixture of pyrimidine (in all of the experiments reported in this paper, the 5-aminomethyl pyrimidine (R = NH_2) was used) and thiazole at different concentrations were harvested after 5 or after 10 days, and the vitamin B_1 in the medium and in the mycelium determined separately by the *Phytophthora* assay. From table 1 it may be seen that no significant amount of vitamin

B_1 was found at the end of 10 days if the initial amount of pyrimidine and thiazole was low. If, however, larger amounts of these substances were added initially, considerable amounts of vitamin B_1 were found at the end of the 10-day experimental period. It is also clear from table 1 that under certain conditions vitamin B_1 is formed and subsequently disappears.

TABLE 1

SYNTHESIS OF VITAMIN B_1 FROM PYRIMIDINE-THIAZOLE MIXTURE BY *Phycomyces*

GROWTH FACTOR SUPPLIED MOLS × 10^{11}	MG. DRY WEIGHT OF *Phycomyces* MYCELIUM AFTER 5 DAYS	10 DAYS	VITAMIN B_1 PRESENT (MOLS × 10^{11}) IN MEDIUM 5 DAYS	10 DAYS	IN MYCELIUM 5 DAYS	10 DAYS
Py + Th* 10	12	15	0.6	0.6	0.6	0.0
" 100	54	76	7.4	1.2	21	3.1
" 1000	58	96	21	34	131	370

*Py = vitamin pyrimidine; Th = vitamin thiazole.

In experiments of a similar type the course of this formation (from pyrimidine and thiazole) and disappearance of the vitamin was followed during a 10-day period. It was found that vitamin was synthesized rapidly after inoculation of the *Phycomyces* culture and that synthesis continued during active growth of the mycelium. A rapid disappearance of the vitamin immediately after cessation of growth was observed, and in numerous experiments the vitamin was found to be virtually or completely absent after 10 days. The points to be stressed in the present connection are (1) that *Phycomyces* does synthesize vitamin from pyrimidine plus thiazole, and (2) that vitamin is destroyed by the resting mycelium.

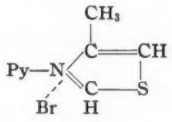
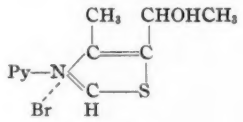
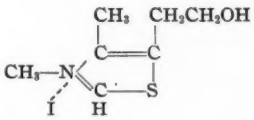
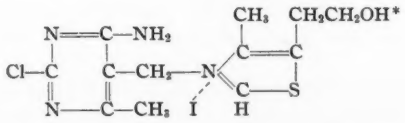
One might next inquire as to the fate of this vitamin which has disappeared. Experiments were made in which *Phycomyces* was allowed to grow for 10 days upon medium containing either vitamin B_1 (10^{-7} molar) or an equivalent mixture of vitamin pyrimidine and vitamin thiazole. At the end of 10 days both mycelium and medium from these cultures were assayed by *Phytophthora*, by *Phycomyces* and by *Phycomyces* in the presence of excess vitamin thiazole. The difference in the values from these two *Phycomyces* assays should be a measure[7] of excess vitamin pyrimidine over vitamin thiazole in the sample under investigation. In the experiment in which vitamin B_1 was added to the culture medium, the assay showed that, after 10 days, the vitamin had completely disappeared from both medium and mycelium. Practically no uncombined thiazole but a considerable quantity of free pyrimidine[8] was found both in the medium and in the *Phycomyces* mycelium. In fact, it was possible to account for, as free pyrimidine, more than half of the pyrimidine added initially as vitamin to the culture. A similar end result was obtained when *Phycomyces* was allowed to grow on the pyrimidine-thiazole mixture.

Since vitamin B_1 is broken down with resultant destruction of the thi-

azole portion, the free pyrimidine liberated should be available for combination with more thiazole. One would expect to find that the ratio of pyrimidine to thiazole optimal for supporting the growth of *Phycomyces* is not 1 to 1 (equimolecular)[4b] but rather that a given number of mols of pyrimidine should suffice for a larger number of mols of thiazole. In numerous experiments it has been found that when the concentration of

TABLE 2

VITAMIN ANALOGS AS SOURCES OF PYRIMIDINE

GROWTH FACTOR SUPPLIED (ALL SUBSTANCES 10^{-7} MOLAR IN THE CULTURE MEDIUM)	MG. DRY WEIGHT OF MYCELIUM PER CULTURE
Py + Th	80
	1
" + Th	70
	0
" + Th	75
	0
" + Py	1
	0
" + Py	2

* Kindly furnished by Professor A. R. Todd.

pyrimidine is small (10^{-8} molar) a ten times excess of thiazole will suffice for at least twice as much growth of mold as will a quantity of thiazole equivalent to the amount of pyrimidine used, whereas excess pyrimidine has no such pronounced effect. Similarly the addition of free thiazole to cultures containing vitamin B_1 itself results in a like marked increase of growth over that obtained with vitamin alone. A pyrimidine molecule

may then be available more than once for combination with vitamin thiazole. Thus the breakdown *in vivo* of the vitamin by *Phycomyces* appears to proceed by a mechanism which leaves the pyrimidine portion intact but which inactivates or destroys the thiazole half.

This mechanism apparently applies also to the case of analogs of the vitamin. Table 2 shows two vitamin analogs made up of the pyrimidine and a thiazole which is an inactive analog of the vitamin thiazole. These vitamin analogs, although themselves inactive, serve as sources of vitamin pyrimidine if the correct vitamin thiazole is present. In contrast to this, two vitamin analogs which are quaternary salts of the vitamin thiazole, but in which the group joined to the thiazole possesses a structure other than that

TABLE 3

DESTRUCTION OF THIAZOLE BY *Phycomyces* MYCELIUM IN THE PRESENCE OF PYRIMIDINE

MYCELIUM EMPLOYED		GROWTH FACTOR ADDED (10^{-7} MOLAR) AT BEGINNING OF PERIOD I	PERIOD II	MG. DRY WEIGHT* OF MYCELIUM PER CULTURE
Mycelium grown on Py + Th (each 10^{-8} molar) for 5 days. Dry weight approx. 9 mg. (pyrimidine containing)	A	0	0	9
		Py	0	9
		Th	0	17
		Py + Th	0	78
	B	Py (for 1 day)	Th	74
		Py (for 5 days)	Th	74
		Py (for 10 days)	Th	78
	C	Th (for 1 day)	Py	72
		Th (for 5 days)	Py	26
		Th (for 10 days)	Py	18
Mycelium from control cultures one month old. (pyrimidine free)	D	0	0	4
		Th	0	4
		Py + Th	0	79
	E	Py (for 5 days)	Th	75
		Th (for 5 days)	Py	81

*Weight taken 10 days after last addition of growth factor.

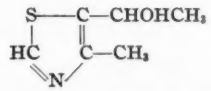
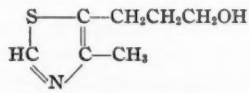
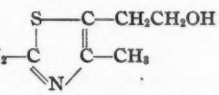
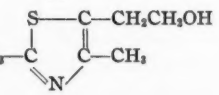
of the vitamin pyrimidine, apparently cannot be utilized at all as sources of thiazole in the presence of vitamin pyrimidine. Thus it seems that analogs of the vitamin are also broken down in a manner such as to make the pyrimidine but not the thiazole portion of the molecule available for recombination.

Next we may consider the question of whether the mechanism of thiazole destruction involves a preliminary dissociation of vitamin into free thiazole; that is, is it thiazole as such which is destroyed or thiazole in the form of a quaternary salt as in the vitamin? The data of table 2 suggest the latter alternative and this view is strongly supported by the evidence presented in table 3. *Phycomyces* mycelia were obtained (as indicated in

the table) either by a method which permitted the retention of a small amount of pyrimidine or by a method which left the mycelium pyrimidine free.[9] Uncombined pyrimidine or thiazole was allowed to remain in contact with these mycelia for varying periods of time as indicated and then the complementary portion of the vitamin was introduced sterilely. It may be seen first (3A) that addition of excess pyrimidine alone exerted no influence on the further growth of the mycelium. The addition of thiazole to pyrimidine containing mycelium brought about a two-fold increase in mycelium dry weight. This is in accord with the fact noted above that a small amount of pyrimidine may suffice for a large amount of thiazole. Uncombined pyrimidine is not affected in its ability to support the growth

TABLE 4

COMPETITION EXPERIMENTS

GROWTH FACTOR SUPPLIED			MG. DRY WEIGHT OF MYCELIUM PER CULTURE
Py + Th (each 10^{-7} molar)			60
" "	+ 	(10^{-5} molar)	58
" "	+ 	"	60
" "	+ 	"	17
" "	+ 	"	10

of *Phycomyces* even though the pyrimidine has been as many as 10 days in contact with mycelium (3B). We must then conclude that pyrimidine is not destroyed at an appreciable rate by resting *Phycomyces*. In marked contrast to this, thiazole is destroyed by contact with pyrimidine containing mycelium (3C) and has completely disappeared after 10 days, since addition of pyrimidine after this time resulted in no more growth than if the pyrimidine were not added at all. On the other hand, destruction of the thiazole apparently does not proceed in the absence of pyrimidine (3D and 3E). Thiazole remained quantitatively intact in these pyrimidine-free cultures even after 5 days of contact with mycelium, whereas in the pyrimidine containing cultures ca. 90% was destroyed after this time. Since

thiazole is destroyed by *Phycomyces* only when pyrimidine is present, we conclude that thiazole is destroyed primarily when it is combined in the form of vitamin.[10]

Further confirmation of this view and some additional insight into the mechanism by which the thiazole ring of the vitamin molecule is broken down is offered by competition[11] experiments. In these experiments *Phycomyces* was grown on a medium containing vitamin pyrimidine, vitamin thiazole and a large excess of inactive thiazole analog.[12] As may be seen in table 4, this large excess of added thiazole analog exerted no significant effect on the growth of the mold provided only that the analog in question was unsubstituted in the 2-position. If, however, the analog was substituted in the 2-position by CH_3 or NH_2, it exerted a marked depressant effect on the ability of the mixture to support the growth of *Phycomyces*. This depressant effect is not due to toxicity of the 2-substituted analogs. In accordance with the above views, we may suppose[13] that in the case of 2-unsubstituted thiazole analogs the corresponding inactive vitamin analog is synthesized *in vivo* but is again broken down, making the pyrimidine eventually available for combination with vitamin thiazole. In the case of 2-substituted thiazoles, on the other hand, the corresponding (inactive) vitamin analog, once synthesized, is not readily broken down, so that pyrimidine, once so bound, is not available for future combination with vitamin thiazole. It may then be postulated that the breakdown of the quaternary thiazole is initiated by a rupture of the ring adjacent to the 2-position, and that the ring is stabilized by the presence of a 2-substituent.

Summary.—Vitamin B_1 is synthesized from a mixture of vitamin pyrimidine and vitamin thiazole by *Phycomyces*. Subsequently the vitamin is broken down by the mycelium with destruction of thiazole and liberation of free pyrimidine. The thiazole portion is attacked presumably only when combined in the form of vitamin (as shown by the fact that its destruction takes place only in the presence of the pyrimidine half). Evidence is presented indicating that the first step in the *in vivo* degradation of the thiamin molecule involves an opening of the quarternary thiazole ring adjacent to the 2-position.

Acknowledgment.—The chemical portion of this work was made possible through a grant from the Research Corporation, for which the authors express their gratitude. The biological testing was carried out with the aid of the Works Progress Administration, Official Project Number 665-07-3-83, Work Project Number 9809.

[1] Bonner, J., and Buchman, E. R., *Proc. Nat. Acad. Sci.*, **24**, 431 (1938).

[2] Bonner, J., and Erickson, J., *Amer. Jour. Bot.*, **25**, 685 (1938).

[3] The additions of vitamin B_1 and/or related substances were made from concentrated stock solutions so that the dilution of the culture medium was negligible.

[4] *a.* Schopfer, W. H., and Jung, A., *Compt. Rend. Acad. Sci. Paris*, **204**, 1500 (1937);

b. Robbins, W. J., and Kavanagh, F., *Proc. Nat. Acad. Sci.*, **23**, 499 (1937); *c.* Sinclair, H. M., *Nature*, **140**, 361 (1937).

[5] It is probable that combined forms of the vitamin such as cocarboxylase are determined by *Phycomyces* and by *Phytophthora* as well. This would not alter the arguments presented here.

[6] Robbins, W. J., *Proc. Nat. Acad. Sci.*, **24**, 53 (1938).

[7] The values for excess pyrimidine obtained in this way are somewhat too high due to the fact that pyrimidine in the presence of excess thiazole may react more than once. In the present experiments, however, this error is sufficiently small to be disregarded.

[8] The work of G. M. Hills (*Biochem. Jour.*, **32**, 383 (1938)) indicates that, also with *Staphylococcus aureus*, the *in vivo* destruction of thiazole takes place more rapidly than that of pyrimidine.

[9] That these mycelia were actually pyrimidine free is indicated by the fact that addition of thiazole promoted no further growth (see table 3D and compare 3A).

[10] Preliminary experiments with *Phytophthora* indicate that this organism (which cannot combine thiazole with pyrimidine) also destroys vitamin with liberation of pyrimidine but does not attack uncombined thiazole.

[11] See reference 1, page 433.

[12] The very slight vitamin thiazole activity of the hydroxypropyl thiazole analog (see reference 2) may be neglected in evaluating the results of table 4.

[13] More direct evidence in support of this interpretation will be presented in a later communication.

TEMPERATURE AND THE CRITICAL INTENSITY FOR RESPONSE TO VISUAL-FLICKER. III. ON THE THEORY OF THE VISUAL RESPONSE CONTOUR, AND THE NATURE OF VISUAL DUPLEXITY

By W. J. Crozier and E. Wolf

Biological Laboratories, Harvard University

Communicated March 7, 1939

I. In most vertebrates, including man, there occur two chief types of retinal photoreceptors, rods and cones. Evidence of various kinds has been very generally held to support the doctrine that the duplexity of the visual performance functions as obtained from the majority of vertebrates tested, including man, is a direct consequence of the specific differences in the excitabilities of retinal rods and cones.[1] Experimental data show unequivocally that, for certain vertebrates, in the production of the duplex performance contours there are indeed concerned the activities of two groups of sensory effects.[2] That the observable properties of the elements of sensorial effect in these 2 groups are determined by the quantitative differentiation of the primary receptors into two categories with respect to the photochemical basis of excitation[1, 3]—or indeed that the quantitative features

of the data are *retinally* determined at all—is, however, much too large an assumption.[4] It can be most conveniently tested, at the moment, by means of observations on response to visual flicker. Considerations arising from the investigation of phenomena of intensity discrimination show that the relation of critical intensity to flash-frequency is of the same form as, for example, the relation of intensity to differential sensitivity ($1/\Delta I$). The argument can therefor be generalized.

The study of flicker-response contours from a considerable diversity of animals shows that the curves of F vs. log I_m are properly described as probability integrals[5] (F = flash-frequency; I_m = mean flash intensity critical for response signifying discrimination of flashes from dark intervals). The data (1) are adequately described by this function, for both rod- and cone-retinas and those containing both, which (2) provides a tested dissection of duplex response contours; and (3) the 3 parameters of this curve exhibit ascertained properties[6] expected on the assumptions leading to the use of this particular equation.

That the general form of the $F - \log I_m$ curve is the expression of a non-specific feature of the data is proved by the facts that (1) it applies to measurements with very different organisms;[5] (2) as well as for dynamically similar situations not involving visual excitation;[5] and in cases where (3) a different physical substratum must be assumed for the mechanism of excitability in the two parts of a duplex flicker-response contour.[7]

The dependence of the excitability mechanism upon temperature should ideally provide a means of analytically separating the rôle of certain obviously statistical factors from that of other, more simple physicochemical influences, in its quantitative expression. When the temperature of the organism is altered, $1/I_m$ (F fixed) obeys the Arrhenius equation as a function of temperature.[8] The *shape* of the flicker contour is independent of temperature, for both "rod" and "cone" portions. This means that the excitability of each neural element involved rises as the temperature rises, so that the intensity I required for its excitation reciprocally declines. The frequency distribution of $d \log I$ for the magnitudes of the sensory effects produced is not changed in form.

II. With the sunfish *Enneacanthus* we find[9] "rod" and "cone" branches of the $F - \log I$ curve which differ in shape constants. Yet the temperature characteristics μ for $1/I_m$ are identical for the two sections of the duplex curve. This is all the more impressive because the Arrhenius plots of $1/I_m$ reveal for this animal a distinctly uncommon type of organization with respect to dependence on temperature. Consequently it cannot be held that the mechanisms of excitability responsible for these 2 parts of the $F - \log I$ curve are chemically different, or essentially different in chemical organization. However, the attempt[10] to deduce the direct photochemical kinetics governing the excitability from the shapes of the components of the

F — log I graph (at one temperature and one light-time ratio in the flash-cycle has been shown[11, 3] to require quite different systems of reaction orders for the "rod" and "cone" segments. If the form of the F — log I graph is a purely statistical matter, this incongruity is naturally of no concern, nor even surprising; nor are the otherwise embarrassing facts that the shape of the curve is independent of temperature and that the dependences of its parameters upon temperature and upon light-time ratio are contrary to those required by a photostationary state formulation.

III. The data have been extended to embrace the properties of the homologous putative "cone"- and "rod"-determined branches of the curves for different vertebrates. Teleosts of the genera *Xiphophorus* (*X.*, sword-tail) and *Platypoecilus* (*P.*, platy) are sufficiently alike in a general chemical sense so that they can produce fertile offspring when cross-bred. Their respective F — log I_m curves are quite unlike.[12] The nearest approach to a photostationary state description[12, 3] of the "rod" curves requires for *X.* and *P.* a second order "dark" reaction, but a first order "light" process for *X.* and a second order for *P.*; with the "cone" segments the light processes for both are required to be taken as of first order, but the dark reaction for *X.* has to be second order, for *P.* first order. The *X.* and *P.* curves differ[12] in maximum ordinates, abscissae of inflection and $\sigma'_{\log I}$. That 2 distinct and separable assemblages of elements are concerned is proved by their independent behavior in inheritance.[12]

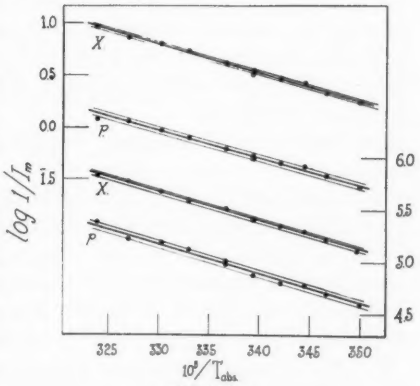


FIGURE 1

Log $(1/I_m)$ vs. $1/T_{abs.}$ for the "cone" (upper) and "rod" (lower) sections of the F — log I_m contour of *Xiphophorus montezuma* (*X.*) and *Platypoecilius maculatus* (*P.*). For *X.*, $F = 25$ and $F = 4$; for *P.*, $F = 25$ and $F = 2.5$ (see text). Each plotted point is the mean of 15 measurements, 3 on each of the same 5 individuals (independent series are given as open circlets). The slopes of the lines drawn give, in descending order, values of μ, in $ln\ (k/I_m) = -\mu/RT$, which are 12,550; 12,430; 12,450; 12,700; the differences in these values are without significance. The relative variation of I_m is characteristically greater for *P.*

Determinations of $1/I$ have been made with *X.* and *P.* at a number of temperatures for (1) a flash-frequency in the "cone" region, uncomplicated by the "rod" effect ($F = 25/sec.$), and (2) in the purely "rod" part of the flicker recognition contour (for *X.*, $F = 4$; for *P.*, $F = 2.5$). The data[13] are plotted in figure 1.

As with *Enneacanthus*,[9] μ for the "rod" and "cone" segments is the same for both *X.* and *P.* It is of particular interest that for these animals the Arrhenius plot exhibits no "break," so that a single μ describes the data over the whole workable range (12.5° to 36°C.).

IV. The temperature characteristics for $1/I_m$ in various kinds of animals[8] are very different. This is in keeping with the fact that turtle, dragon fly nymph, sunfish and swordtail clearly are in a chemical sense unrelated systems. Swordtail *X.* and platy *P.* must be assumed more nearly alike. The forms of the $F-\log I$ curves for these various animals are very definitely specific. Consequently it could not be decided with the data on such animals as, for example, turtle and sunfish, whether the differences in shape constants are really to be correlated with differences in the chemical kinetics of excitability.

But with swordtail and platy this question can be answered. The $F-\log I$ curves differ in every measurable feature. That they are determined by physical realities of organization in the two forms is proved (1) by the fact of reproducibility, (2) by the quantitative consistency of performance in various species and varieties of each genus[2] and especially (3) by the behavior of the shape constants in breeding experiments.[12] The essential identity of μ (Fig. 1) for *X.* and *P.*, however, forbids the assumption that the processes governing the respective excitabilities are kinetically different.

V. The necessary conclusion is that the chemical kinetics of excitation cannot be deduced from the shape of the performance curve as a function of intensity. The form of the curve is a statistical consequence of the fact that a kinetically homogeneous population of excitable units with thresholds forming a frequency distribution of $d(1/I)$ gives rise to a population of mean sensory effects which is normally distributed in terms of $\log I$. The specific invariant *shape* parameters of this distribution have no observable connection with peripheral reaction velocities implicated in excitability. Rise of temperature merely lowers the intensity threshold for each of the excitable units, which are a homogeneous group with respect to chemical organization. Change of temperature does not alter the total number of units of sensory effect, and does not change the shape of their frequency distribution.

Duplexity of visual performance contours in vertebrates is due to differences in the statistical properties of 2 sets of elements of neural effect, and to differences in their mean excitabilities. These differences do not involve a differentiation of organization with respect to chemical reaction kinetics underlying excitability. They are readily understood as due to differences in total numbers of elements and the presence of different amounts of catalyst in the excitability process; or even as a physical consequence of the presence of a photic sensitizer in connection with one group of elements. The reaction velocity governing excitability shows by its

different temperature characteristics in various animals, and by its identity of μ for the two sets of elements of sensory effect ("rod" and "cone") in any one animal, that the controlling chemical processes in the two sets cannot be assumed to be kinetically different.[14]

The behavior of hybrids of *X.* and *P.* gives additional evidence on this matter.

[1] For an able summary of data and deductions from this standpoint, cf. Hecht, S., *Physiol. Rev.*, **17**, 239 (1937).

[2] Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **21**, 17 (1937–1938); *Ibid.*, **22**, 311 (1938–1939); *Proc. Nat. Acad. Sci.*, **23**, 516 (1937); *Ibid.*, **24**, 221 (1938); Crozier, W. J., and Wolf, E., *Ibid.*, **24**, 542 (1938); *Jour. Gen. Physiol.*, **22** (in press) (Mar., 1939) (1938–1939); Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Proc. Nat. Acad. Sci.*, **24**, 125, 538 (1938); *Jour. Gen. Physiol.*, **22** (in press) (May) (1938–1939).

[3] Hecht, S., *Jour. Appl. Phys.*, **9**, 156 (1938); The Harvey Lectures, **1937–1938**, 35 (1938).

[4] Crozier, W. J., and Wolf, E., *Proc. Nat. Acad. Sci.*, **24**, 538 (1938); *Jour. Gen. Physiol.*, **22** (in press) (May) (1938–1939); Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Proc. Nat. Acad. Sci.*, **24**, 125 (1938); Crozier, W. J., *Ibid.*, (in press) (1939); Crozier, W. J., and Holway, A. H., *Jour. Gen. Physiol.*, **22**, 351 (1938–1939).

[5] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **23**, 71 (1937); Crozier, W. J., Wolf, E., and Zerrahn-Wolf, G., *Proc. Nat. Acad. Sci.*, **23**, 516 (1937); *Ibid.*, **24**, 216 (1938); *Jour. Gen. Physiol.*, **22**, 311 (1938–1939); *Ibid.* (in press); Crozier, W. J., and Wolf, E., *Proc. Nat. Acad. Sci.*, **24**, 542 (1938).

[6] *Jour. Gen. Physiol.*, **20**, 393, 411 (1936–1937); *Ibid.*, **22** (in press) (1938–1939); *Ibid.*, **21**, 223, 313, 463 (1937–1938).

[7] *Jour. Gen. Physiol.*, **21**, 17 (1937–1938); *Ibid.*, **22**, (in press) (March) (1938–1939); *Proc. Nat. Acad. Sci.*, **23**, 516 (1937); *Ibid.*, **24**, 542 (1938).

[8] *Proc. Nat. Acad. Sci.*, **24**, 216 (1938); *Ibid.*, **25**, (in press) (1939); *Jour. Gen. Physiol.*, **22**, 311 (in press) (1938–1939).

[9] Crozier, W. J., *Proc. Nat. Acad. Sci.*, **25**, (in press) (1939); Crozier, W. J., and Wolf, E., *Jour. Gen. Physiol.*, **22**, (in press) (March) (1938–1939).

[10] Hecht, S., Shlaer, S., and Smith, E. L., *Cold Spr. Harb. Symposia Quant. Biol.*, **3**, 237 (1935).

[11] Crozier, W. J., Wolf, E., Zerrahn-Wolf, G., *Jour. Gen. Physiol.*, **20**, 393 (1936–1937).

[12] *Jour. Gen. Physiol.*, **21**, 17 (1937–1938); *Ibid.*, **22** (in press) (March) (1938–1939); *Proc. Nat. Acad. Sci.*, **23**, 516 (1937); *Ibid.*, **24**, 221, 542 (1938).

[13] A more complete account of these observations will be given elsewhere. We are indebted to Dr. Gertrud Zerrahn-Wolf for assistance in the experiments. Thermostatic equipment was in part provided through a grant to one of us by the Trustees of the Elizabeth Thompson Science Fund, to whom our thanks are due.

[14] It is to be noted that there are certain facts entirely consistent with the direct application of these considerations, which on the alternative view require the assistance of subsidiary hypotheses: thus photic adaptation of a retina may occur without detectable change in the rod visual purple (Granit, R., *Jour. Physiol.*, **94**, 430 (1938)); and "rod" and "cone" visual thresholds during dark adaptation under modifications of vitamin *A* intake show changes which are essentially parallel (Haig, C., Hecht, S., and Patek, A. J., Jr., *Sci.*, **87**, 534 (1938); Hecht, S., and Mandelbaum, J., *Sci.*, **88**, 219 (1938); Wald, G., Jeghers, H., and Arminio, J., *Amer. Jour. Physiol.*, **123**, 732 (1938)).

SPECIFIC CONSTANTS FOR VISUAL EXCITATION. IV. ON THE NATURE OF GENETIC DIFFERENCES

BY W. J. CROZIER AND E. WOLF

BIOLOGICAL LABORATORIES, HARVARD UNIVERSITY

Communicated March 7, 1939

I. The invariant organic properties denoted by various specific constants for visual excitation,[1] derived from measurements of visual performance as a function of intensity, present certain opportunities for genetic analysis.[2] Genetic theory has developed in two rather distinct directions. In one, an abstract conceptual mechanism has been derived embracing laws describing the manner in which segregations of the determinants of heritable characteristics behave in breeding experiments. In the other, chromosomal structures have been identified which in minute detail parallel as to geometrical pattern and in segregating behavior the known distributions of heritable features. The mathematical theory of breeding results is not necessarily dependent on the chromosomal localizations of point mutations, but the desire to unify these two groups of data has had certain consequences for genetic theory. It has led to attempts to obtain some notion of the means whereby in the course of development quite minute particles of chromosomal substance, in specific positions and in particular architectural relationships with a large number of others, are enabled to be responsible for definite and stable differences in observable features of the organism.

Speculations of this order have not, however, really answered the question sometimes phrased in the form "What *is* a gene?" One of the most important reasons for this is that the label "gene" has been used for several different kinds of abstractions. In one sense it has been applied to those paired determining entities, differences between which are responsible for recognizable inherited differences. A theory of the nature of genetic differences does not in fact necessarily lead to or depend upon a conception of the intimate nature of the gene itself. The temptation has been insistent, however, to directly link those physicochemical properties which must be ascribed to the chromosomal gene with the mechanism controlling the observable characteristic for which it is taken to be responsible. Among the essential dynamical properties of the chromosomal gene are its catalytic capacity for specific propagation and the automatic regulation of this capacity. Translations of such properties into explanations for specific resultants in development, whether or not they involve obvious participation of enzymes, is, however, faced with a serious theoretic difficulty. In many cases the genetically significant alternative properties of an organism cannot be described in terms of mere presence or absence with respect to

some prevalent standard. They are to be recognized as *capacities* for organic performance. As such they can be rationally described and measured only in terms of their quantitative dependence upon significant independent variables.[3] A sufficient variety of such instances has been established empirically[2-4] to justify expectation that many others of the same formal type will in time be found. The difficulty these cases present arises from the fact that in the expression of performance properties we have commonly to do with the simultaneous or integrative activities of groups of cellular units. The statistical properties of such assemblages unquestionably play a rôle in the quantitative features of performance. It is required to identify separately the share of such statistical properties. Until this is done theory cannot deal with those common features of chemical organization which permit such cellular units to behave as a homogeneous population in the control of a characteristic property known to be determined in a genetically simple way.

II. This general problem is of a kind fundamental for many sorts of physiological analysis. It requires to be shown how an assemblage of units, differing statistically among themselves, can with respect to certain variables provide measures of performance which adhere to comparatively simple physicochemical rules. It is sometimes supposed that this may come about because the variable contributions of each active element "average out" in a uniform way. But this involves so many assumptions as to amount to a mere evasion of the problem.

The properties of flicker-response contours found in crosses of two teleost genera illustrate the nature of the problem in some detail. The essential points[5] are: (1) The form of the $F - \log I_m$ curve for response to visual flicker differs in *Xiphophorus* and *Platypoecilius* in a number of quantitative features, just as do the corresponding curves for other genera. (2) In hybrids of *X.* and *P.* there is manifested a simple dominance of certain quantitative properties of the curves, since particular parameters of the curve for the hybrids agree precisely with those characteristic of one or the other parent stock. Analysis shows that the "rod" and "cone" portions of each of the duplex $F - \log I$ contours represent summations of neural effects produced in or by two physically distinct populations of elements; each of them is in form a probability integral, with different values for their 3 parameters.

The simplest approach to the mensuration of relevant kinetic chemical properties of such groups of elements is given by estimations of their apparent energies of activation for threshold excitation. It turns out that for *Xiphophorus* and *Platypoecilius* the temperature characteristics μ for both "rod" and "cone" branches of the curves are the same.[6] Hence it cannot be assumed that the differing shapes of the $F - \log I$ contours are due to differences in the chemical organization of excitability, in either the

"rod" and "cone" sets of elements or in the individuals of the 2 genera. The forms of the curves, their shape constants, are due to the operation of statistical attributes of the assemblage of organic units concerned, which measure the capacity for visual performance under the given conditions. Yet the specific shape constants are definitely heritable.[5]

III. In the F_1 progeny of *Xiphophorus* × *Playpoecilius* the "rod" portion of the $F - log\ I$ curve is intermediate[5] between those for *X.* and *P.* in its position on the log I scale, and in its value of F_{max}. This, one sup-

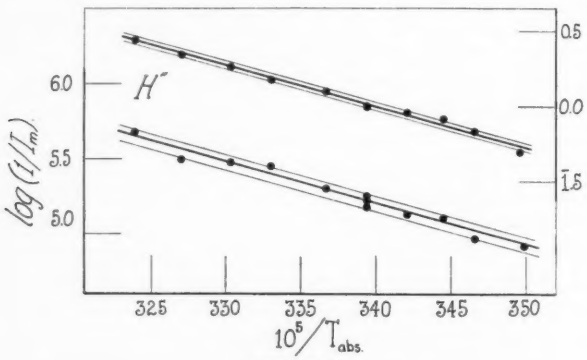


FIGURE 1

As the temperature is increased, the excitability of response to visual flicker (measured by $1/I$, with F constant) increases. For *Xiphophorus* and *Platypoecilius*[6] the increase is described by the Arrhenius equation, with $\mu = 12{,}400 \pm$, between 12.5° and 35.6°, and μ is independent of the flash-frequency. For the F_1 hybrids of *X.* and *P.* the same rule is obeyed, with the same value of μ (upper line, $\mu = 12{,}400$; lower, $\mu = 12{,}600$). The upper curve refers to the "cone" segment of the $F - log\ I$ contour ($F = 25$ *per sec.*), the lower to the "rod" part ($F = 3$). The shape constants for the "cones" and "rod" segments differ markedly in *X.* and *P.*, and the other parameters as well. For F_1 the shape constant ($\sigma'_{log\ I}$) for the upper curve agrees with that for *X.*, while for the lower ("rod") part it agrees with that for *P.*

poses, could be taken as an instance of "blending." But its shape constant is that for *P.*, and μ for $1/I$ (with F fixed) is the same as for *X.* and *P.* (Fig. 1). The "cone" portion of the graph for F_1 has quantitatively[5] the form of that for *X.* although situated at a higher level on the log I axis; as figure 1 shows, μ is again the same.

IV. These results appear to demonstrate that statistical features of organization can be heritable, without the occurrence of admissable differences of chemical organization as a determinant of the observed genetic difference. By "statistical features" we here refer, of course, to those

properties of the units in a set or group which arise from the form of the random distribution and the random variation of capacities to contribute to the determination of the observed and measured result. The mere positions of the parts of the $F - log\ I$ curve on the log I axis could be accounted for in terms of various average amounts of substance involved in the excitation process, but it is not these positions on the coördinate grid which provide interpretable indices of hereditary transmission of the visual performance functions. Such indices are given by the shape parameters, which are statistical properties.

If this situation were to be examined after the procedure customary in breeding experiments, we could seek to compare $X.$, $P.$ and F_1 under a certain uniform condition. Suppose that a particular value of F were chosen. Above $F = 40$ response to flicker could be obtained[5] from $P.$, but not from $X.$ or F_1 (without increase of the dark time ratio in the flash cycle). At a somewhat lower F the $X.$ and $P.$ curves cross, and the critical intensity for F_1 is higher than for either. A variety of other relationships would appear if other values of F or I were to be taken as the standard experimental condition. Only on the basis of the entire function can the relations between the performance curves for $X.$, $P.$ and F_1 really be seen or interpreted theoretically. The formal point is not new,[3] but the present instance of its usefulness seems to provide a basis for interesting extensions of its application.[7]

[1] *Proc. Nat. Acad. Sci.*, **23,** 516 (1937); **24,** 221 (1938); **24,** 542 (1938); *Jour. Gen. Physiol.*, **21,** 17 (1937); (in press) (1939).

[2] *Proc. Nat. Acad. Sci.*, **23,** 516 (1937); **25** (in press) (1939); *Jour. Gen. Physiol.*, **21,** 17 (1937); (in press) (1939).

[3] *Proc. Nat. Acad. Sci.*, **12,** 612 (1926); *Jour. Gen. Physiol.*, **13,** 57, 81 (1929); *Proc. 6th Intern. Congr. Gen.*, **2,** 31; *Détérminisme et Variabilité*, Paris, Hermann et Cie, 1935, 57 pp.

[4] *Jour. Gen. Physiol.*, **15,** 437 (1932); **20,** 111 (1936); Pincus, G., *Jour. Gen. Physiol.*, **14,** 421 (1931).

[5] *Jour. Gen. Physiol.*, **21,** 17 (1937); (in press) (1939).

[6] *Proc. Nat. Acad. Sci.*, **25** (in press) (1939).

[7] A detailed account of the observations will be given elsewhere.

For one of the thermostats used we are under obligation for a grant from the Trustees of the Elizabeth Thompson Science Fund.

A NOTE ON POLARITY POTENTIALS FROM THE HUMAN EYE

BY DONALD B. LINDSLEY AND WALTER S. HUNTER

EMMA PENDLETON BRADLEY HOME AND BROWN UNIVERSITY

Communicated March 15, 1939

Because of the renewed interest in the polarity potentials of the human eye,[1, 2, 3] it seems worth while to insert in the literature records of such potentials secured by simultaneous multiple recording from four pairs of electrodes. The records here published support very conclusively the view that such potential differences arise from the polarity of the eye (negative in the back of the bulbus and positive in the front) rather than from the admitted activity of the extra-ocular muscles in turning the bulbus and in maintaining the resulting fixations. The records reveal muscle action potentials; but the great swings in potential do not arise from such a source.

Small chlorided silver hat-like electrodes with an inside diameter of 5 mm. and resting on felt brims were attached to the surface of the skin by means of collodion. Contact with the skin was maintained by means of electrode jelly which was injected through a small hole in the dome of the electrode hat. The electrodes were placed as closely as possible to the eyes but over the stationary bony margins of the orbits. Four electrodes were placed about the right eye on the nasal, temporal, infra- and supra-orbital (above eyebrow) margins. In addition, an electrode was attached to the temporal margin of the left eye. Potential changes were recorded simultaneously from four paired combinations of the five electrodes as follows: (1) right temporal-right nasal (horizontal leads); (2) right infra-orbital and right supra-orbital (vertical leads); (3) right temporal-left temporal (temporal leads); and (4) right supra-orbital and right nasal (oblique leads). The method of bi-polar recording with none of the leads directly grounded was used.

Four independent amplifying channels and a Westinghouse, type PA, four element oscillograph made possible the simultaneous recording from four pairs of electrodes. The amplifiers were of the push-pull type throughout and were coupled with 2 microfarad condensers and 0.5 megohm grid resistors which provided a time constant of the order of one second. Direct current potentials of the type dealt with in this study show therefore only an initial surge of the recording line with each eye movement and then a gradual drift back to the baseline. The magnitude of the potential change between any two electrodes with each sudden eye movement is represented by the initial deflection. (If one is interested in recording rapid eye movements as relatively discrete phenomena, it is perhaps better to use an amplifier with a shorter time constant. Since 0.5 and 0.1 microfarad con-

denser coupling values are interchangeable in our amplifiers by a simple switching arrangement, we have also used the shorter time constant values in other recordings not included in figure 1. In that case there was a quicker return of the recording line to the baseline after each eye movement than is the case in figure 1.)

Figure 1 presents the relevant findings of the present study. The bottom line in each record gives time in seconds. The lines above show, re-

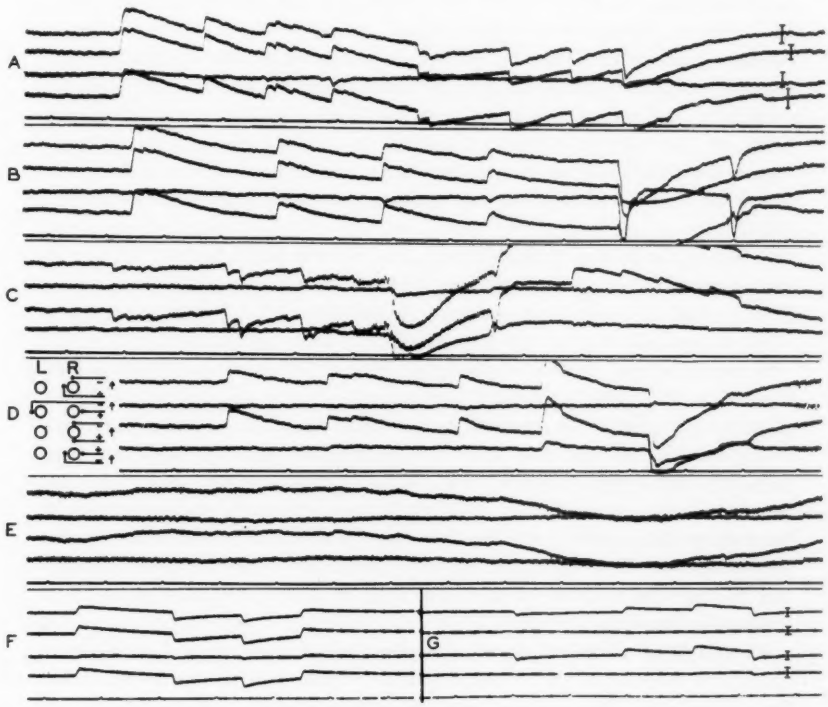


FIGURE 1

spectively, potentials from the (1) horizontal, (2) vertical, (3) temporal and (4) oblique leads. The insert at Record *D* is a key for all records. *L* and *R* stand for left and right eye, respectively. The arrows indicate the direction of deflection of the recording lines where the potential differences of the various leads are of the order indicated by the plus and minus signs.

In Record *A*, the subject fixated straight ahead in the horizontal plane (called center fixation), and then fixated four points in succession to the right, holding each fixation momentarily. Return to center fixation was

by the same four stages in reverse. Only the pair of vertical electrodes *failed* to pick up potential differences. Where the potential records are deflected upward, the electrode nearest the back of the eye is negative with respect to its corresponding electrode. Thus as the bulbus turns to the right, the nasal electrode is negative with respect to (*a*) the right temporal electrode and (*b*) the supra-orbital electrode. And the left temporal is negative with respect to the right temporal electrode. The vertical pair, however, being equally near the back of the eye, are unaffected. (As fixation returns from extreme right to center, the polarity of the electrodes is necessarily reversed since, for example, the right temporal electrode is becoming *less* positive in relation to the nasal electrode at each step back to center.) Particularly in the third lateralward and the first medialward fixation of this record, it will be noted that the recording is sufficiently sensitive to show the slight irregularities of fixation which occur.

Record *B* was secured under the same conditions as Record *A* except that fixations were held longer and the return from extreme right to center fixation was made in one quick eye movement. The latter part of Record *B* shows the effect of a blink of the eyelids. The potential differences here were picked up by the vertical and oblique pairs of electrodes since they alone were so placed as to record such effects advantageously.

Record *C* shows the results when the eyes fixated successively four points in the vertical meridian starting from center, going up and returning to center. Here only the vertical and oblique electrodes register potential changes. In each case it is either the lower or the nasal electrode which is negative with respect to the supra-orbital member of the pair. At the fourth (upward) fixation the record shows marked muscle action potentials from the vertical and oblique leads. These may be due to the action of the superior rectus and inferior oblique muscles in rotating the bulbus upward, but they are more probably due to the activity of the muscles of the eyelids (orbicularis oculi and levators).

Record *D* shows the effects when fixation shifts from center down by four successive stages and then returns quickly to center fixation. Again only the vertical and oblique leads reveal potential changes. And it is important to note, in harmony with the polarity of the eye, that these changes are represented by deflections in the opposite direction from those found in Record *C* where the bulbus turned upward. (The key at the beginning of Record *D* has been commented on above.)

Record *E* shows slow swings in potential differences when the eyes turn slowly down and then back to center. Such slow steady movements were secured by having the subject fixate a slowly moving object. Again it is important to note that only the vertical and oblique electrode pairs are affected.

Record *F* shows the result of the following procedure: fixate center, turn

eyes to first right fixation point, hold, return to center and hold, turn to first left fixation and hold, then return to center. The conditions for securing Record *G* were the same except that the shift of fixation was up, center, down and return to center. The potential changes are again entirely consistent with the polarity of the bulbus. (These recordings in *F* and *G* were made with lower amplification than were those in the other records.)

In the records of figure 1 the calibration lines at the right on Record *A* represent 100 μv. This calibration also applies to Records *B*, *C*, *D* and *E*. The calibration lines on Record *G* apply also to Record *F* and represent 200 μv. The magnitude of the potential differences from individual changes in fixation, in the present study, range from about 40 to 400 microvolts, depending upon the extent of the eye movement. The potentials recorded are roughly proportional to the angular deviation of the pupil from center fixation. This holds for fixations along the vertical as well as along the horizontal meridian.

Inasmuch as all of the records of figure 1, as well as others not here included, show deflections consistent with the view that the back of the bulbus is negative with respect to the front and inasmuch as these deflections are of an entirely different character from the muscle action potentials which show in some of the records, figure 1 definitely supports the existence of an electrical polarity of the eye. Furthermore, unpublished observations by one of us, Lindsley, using direct recording of action potentials of the exposed human extraocular muscles indicate that such muscle potentials are of the typical rapidly fluctuating type. The magnitude of such potentials, even from the exposed muscles, is not so great as those here recorded from the margins of the orbit. Although such polarity effects as we here confirm have been termed corneo-retinal potentials,[3] no satisfactory explanation of their origin has yet been established. In arriving at such an explanation, it is not unreasonable to suggest that correlations might be found with such work as that of Burge and his co-workers[4] on the potential differences between the cortex and the sciatic nerve.

[1] Hoffman, A. C., Welman, B., and Carmichael, L., *Jour. Exptl. Psychol.*, **24,** 40–53 (1939).

[2] Miles, W. R., *Proc. Nat. Acad. Sci.*, **25,** 25–36 (1939).

[3] Mowrer, O. H., Ruch, T. C., and Miller, N. E., *Amer. Jour. Physiol.*, **114,** 423–428 (1936).

[4] Burge, W. E., Wickwire, G. C., Orth, O. S., Neild, H. W., and Elhardt, W. P., *Amer. Jour. Physiol.*, **116,** 19–20 (1936).

LEAF GROWTH FACTORS II—THE ACTIVITY OF PURE SUBSTANCES IN LEAF GROWTH

By David M. Bonner and A. J. Haagen-Smit

William G. Kerckhoff Laboratories of the Biological Sciences, California Institute of Technology

Communicated March 14, 1939

It was shown by Went[7] that leaf growth in etiolated pea seedlings is primarily controlled by substances supplied from the cotyledons. Vein growth is controlled by auxin,[2] while growth of the mesophyll is independent of auxin, and it is the factors concerned in the growth of the latter which we will discuss in this paper. In a recent paper[3] a bio-assay for these factors was described in detail, as well as a variety of sources, and so will be discussed only briefly here. An outline of the leaf test is as follows: Circular discs ca. 19.5 mm.^{2} in area are cut from young first foliage leaves of *Raphanus*. Twenty such discs are floated on the solutions to be tested, and allowed to grow on these solutions for 30 hours at 25°C. The total wet weight of all 20 sections from a single solution is then determined by direct weighing in a standard manner. All solutions to be tested are made up to contain 1% sucrose, and activities are compared to the growth of sections grown in 1% sucrose alone. The source from which a solution of standard activity (S. S. A.) is made is the pea diffusate described by Kögl and Haagen-Smit.[6] The present paper will discuss only the activity of various pure substances in increasing mesophyll growth.

TABLE 1

Activity of Amino Acids in the Leaf Test Using Raphanus Leaves As Test Objects. Activity on Basis of Growth in 1% Sucrose Solution Is Equal to 100

Active amino acids	Concentration expressed as mg. per cc. of 1% sucrose solution			
	0.5	0.1	0.02	0.005
Proline	124	114	107	105
Asparagine	116	112	103	101
d-Valine	112	105	103	100
Glutamic acid	toxic	108	102	100
Alanine	110	100	100	100
Leucine	110	100	100	100

Amino acids tested between the concentrations of 0.5–0.005 mg. per one cc. of 1% sucrose solution and found to be inactive:

Histidine, glycine, aspartic acid, arginine, citrulline, serine, phenyl-alanine, cystine and beta-alanine.

In table 1 are listed the amino acids that have been tested in the leaf test. When radish leaves are used as test objects asparagine and proline proved the most active. However, in the case of the amino acids it seems

likely that the species of plant from which the leaf is taken determines the amino acid requirements; thus when *Nicotiana sylvestris* is used as the test object arginine proved to be the most active, while with *Raphanus* leaves arginine was totally inactive. However, the growth obtained by the use of amino acids is never as great as that obtained for the S. S. A., nor are the amino acids active by themselves at high dilution. Table 2 lists a variety

TABLE 2

SUBSTANCES TESTED BETWEEN THE CONCENTRATIONS OF 0.5–0.005 MG. PER CC. OF 1% SUCROSE SOLUTION, AND FOUND TO BE INACTIVE IN THE LEAF TEST USING *Raphanus* LEAVES AS TEST OBJECTS

Nicotinic acid	Vitamin B_1
Nicotinic acid amide	Vitamin B_2
Indole (3) acetic acid	Vitamin B_6
Theelin I	Vitamin E
Inositol	2-methyl, 6-amino pyrimidine
Biotin	2-methyl, 6-hydroxy pyrimidine
Boric acid	Thiazole of vitamin B_1
	Uracil

of substances known to be active in different biological phenomena which have been tested in the leaf test. In every instance they have been found to be inactive with the exception of nicotinic acid amide which showed slight activity at a rather high concentration. Yeast nucleic acid was found to possess very definite activity. Upon testing of various crystalline purines, adenine proved to be active at the highest dilution, 20 gamma per liter, hypoxanthine had activity though at a somewhat higher concentration, while guanine, xanthine, uric acid and caffeine had slight activity or were totally inactive (table 3).

TABLE 3

ACTIVITIES OF PURINES IN THE LEAF TEST USING *Raphanus* LEAVES AS TEST OBJECTS. ACTIVITY ON BASIS OF GROWTH IN 1% SUCROSE SOLUTION EQUAL TO 100

	CONCENTRATION EXPRESSED AS MG. PER CC OF 1% SUCROSE SOLUTION				
PURINE TESTED	0.5	0.1	0.05	0.01	0.005
Adenine	...	...	118	117	113
Hypoxanthine	...	109	116	104	100
Xanthine	...	115	107	100	100
Caffeine	102	112	108	100	100
Uric acid	118	108	100	100	100
Guanine	100	100	100	100	100

That adenine, or a related purine, is the only factor concerned in mesophyll growth seems unlikely not only from chemical studies of the pea diffusate to be reported later, but also from the fact that in only one or two instances was the amount of growth obtainable from leaf sections in adenine plus 1% sucrose as great as that obtainable from the S. S. A. This means,

as shown in table 4, that the amount of growth at the optimum concentration of adenine was not as great as that obtainable at the optimum concentration of the S. S. A. Due to this definite activity it was deemed of interest to test its activity in a variety of ways.

The first type of experiment was that of culturing entire etiolated pea seedling leaves. Etiolated pea seedlings were germinated and grown sterilely in test tubes on nutrient agar. Ten days after germination the leaves, which were very small and curled, were cut off and transferred by sterile technique to 50 cc. Erlenmeyer flasks. The basic inorganic medium used was that used in the culture of pea roots,[4] 20 cc. of medium being used per flask. The following series of cultures were used: A—water; B—inorganic medium plus 1% sucrose; C—B plus 2 mg./l. of adenine; D—B plus 0.2 mg./l. of adenine; E—B plus 200 mg. dry weight of pea diffusate per liter. Ten flasks, with three leaves per flask, were used for each series. After five weeks, observations on the growth in the various solutions were

TABLE 4

COMPARISON OF THE ACTIVITIES OF ADENINE AND PEA DIFFUSATE IN THE LEAF TEST USING *Raphanus* LEAVES AS TEST OBJECTS, WITH COMPARISON ON SEVERAL DIFFERENT DAYS

Activity on Basis of Growth in 1% Sucrose Solution Is Equal to 100.

DATE OF TEST	ACTIVITY AT OPTIMUM CONCENTRATION OF PEA DIFFUSATE	ACTIVITY AT OPTIMUM CONCENTRATION OF ADENINE
November 5, 1938	125	118
November 9, 1938	135	125
November 12, 1938	113	100
November 14, 1938	130	110
December 1, 1938	118	107
January 13, 1939	118	108

made. A concentration of 2 mg./l. of adenine proved inhibitory, while the growth of the leaves in the flasks to which adenine had been added at a concentration of 0.2 mg./l. was greater than in the series with only inorganic medium plus sucrose, though in no instance was the growth as great as that in pea diffusate.[3] Thus in this test the behavior of adenine is similar to that in the leaf test.

The effect of adenine upon the growth of isolated pea roots was also investigated.[8] The method of culturing isolated roots has been reported in detail elsewhere,[4] so will not be described here. In the first six transfers the addition of adenine produced no increase in growth above that of roots growing on vitamin B_1 and nicotinic acid.[1] After six transfers, i.e., after six weeks, adenine began to show some beneficial effect, this effect becoming greater with increase in number of transfers. Although one can conclude that adenine may be a growth factor for pea roots under our conditions of

culture, the effect produced is much smaller than that of other factors as vitamin B_1 and nicotinic acid.[1]

The third type of experiment carried out with adenine was a determination of its effect on *Cosmos* plants grown in the greenhouse in washed sand and supplied with nutrient solution under conditions similar to those of Bonner and Greene.[5,8] Shive's nutrient solution was used, with adenine added to it in the concentrations of 0.5 mg./l., and 0.1 mg./l., each pot receiving 250 cc. of nutrient on alternate days. One pot was used for each

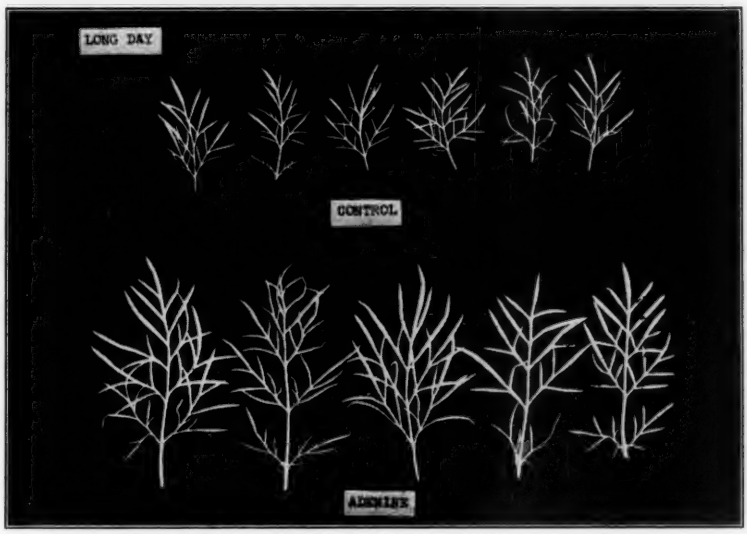


FIGURE 1

Leaves from plants grown with and without adenine.

Control: leaves from *Cosmos* plants grown in washed sand and watered with Shive's nutrient solution, five weeks after germination.

Adenine: leaves from *Cosmos* plants grown under identical conditions but with the addition of 0.1 mg./l. of adenine to the nutrient solution.

of these concentrations, with two pots for controls, and similar series were run on both short and long photoperiods. No appreciable effect appeared during the first two to three weeks after germination. After three weeks the plants treated with 0.1 mg./l. of adenine became steadily larger than the controls with markedly larger leaves. A concentration of 0.5 mg./l. proved in this case to be inhibitory. The effect on size of the plants was much greater for plants on a long photoperiod, with the effect on leaf size being more pronounced in plants growing on a short photoperiod. Figure 1 shows a comparison of leaf size between control plants and treated plants

(0.1 mg./l. of adenine) five weeks after germination. Each leaf was taken from a separate plant, the position of the leaf on the plant being in each case the same. Thus it appears that the addition of adenine to plants can under certain conditions bring about an increase in leaf area.

Discussion.—From a study of the activity of pure substances in the leaf test, one general class of compounds appears to have activity, namely, the purines, with adenine possessing the highest activity of the purines tested. The naturally occurring purine active in leaf growth must await isolation and identification, this work being in progress. It should be emphasized that factors other than adenine, or any single purine, are involved in leaf growth, as was pointed out earlier in this paper. Therefore, it is not surprising that the growth of excised etiolated pea seedling leaves, as has been shown, is not very great. The leaves themselves have but little of these other factors, as shown by the large response to pea diffusate,[3] and the culture conditions are such as to largely preclude their synthesis. They are not furnished with possibly essential amino acids, nor are the light conditions necessarily correct. However, when one gives an excess of one factor to a plant growing normally in a greenhouse, the conditions are considerably different. By increasing the growth rate of the leaves under normal conditions to an extent shown by adenine in the leaf test, the other necessary factors will be formed in greater amount, with the result that the growth rate of the leaves may be further increased. The increase in leaf surface can adequately account for the enhanced growth of the plant as a whole, not only by increasing carbohydrate synthesis but also by increasing phytohormone synthesis. For example, auxin, a stem growth hormone, is known to be produced in green leaves, so that its production may be increased. Vitamin B_1, a root growth hormone, is also known to be synthesized in green leaves. Thus the increase in leaf surface would indirectly cause increased stem and root growth, the resultant plant being normally proportioned, but of larger size and increased general vigor.

[1] Addicott, F. T., and Bonner, James, *Science*, **88**, 577 (1938).

[2] Avery, G. S., Jr., *Bull. Torrey Bot. Club*, **62**, 313 (1935).

[3] Bonner, David M., Haagen-Smit, A. J., and Went, F. W., *Bot. Gaz.* (in press).

[4] Bonner, James, and Addicott, F., *Bot. Gaz.*, **99**, 144 (1937).

[5] Bonner, James, and Greene, J., *Bot. Gaz.*, **100**, 226 (1938).

[6] Kögl, F., and Haagen-Smit, A. J., *Zeit. physiol. Chem.*, **243**, 209 (1936).

[7] Went, F. W., *Plant Physiol.*, **13**, 55 (1938); *Ibid., Am. Jour. Bot.*, **25**, 44 (1938).

[8] The authors are indebted to Dr. James Bonner for these experiments.

[9] The authors wish to thank Mr. L. E. Castle, Jr., for aid in the carrying out of the leaf tests.

[10] Report of work carried out with the aid of the Works Progress Administration Official Project Number 665-07-3-83, Work Project W-9809.

CENTRIOLES AND CHROMOSOMES IN THE ATYPICAL SPERMATOGENESIS OF VIVIPARA

BY ARTHUR WAGG POLLISTER

DEPARTMENT OF ZOÖLOGY, COLUMBIA UNIVERSITY

Communicated March 9, 1939

It has been shown that in many genera of prosobranchiate snails two types of sperms are produced. One of these, the typical or eupyrene, is normal in structure and development; in no way different from that characteristic of Mollusca in general. The other, the atypical, is usually larger, and is very abnormal; the number of axial filaments is increased and the chromatin is either absent (apyrene condition) or greatly reduced in amount (oligopyrene condition). In Vivipara (Paludina) the atypical spermatozoa are oligopyrene. The development of these was worked out in great detail by Meves,[1] 1902, in *V. vivipara*. Considering the remarkable nature of his observations it is surprising that the work has attracted little general attention, being cited chiefly as an example of independent multiplication of centrioles that supports the Henneguy-Lenhossék hypothesis of the nature of the basal bodies of cilia. The fact that in all cases of atypical gasteropod spermatogenesis the increase in number of centrioles is accompanied by very abnormal meiotic behavior and the eventual disintegration of a large number or all of the chromosomes has received little notice. The author is now studying atypical spermatogenesis in an unidentified species of Vivipara (not the same as that studied by Meves) with the object of comparing the behavior of centrioles with that of chromosomes. The results to date confirm all the main features of Meves's account, and they furthermore reveal a strikingly definite numerical relationship between centrioles and chromosomes, the possibility of which is vaguely suggested by Meves's data. The phenomena are so remarkable and are of such general significance that the following brief note seems warranted in advance of publication of full details.

In this species the diploid number of chromosomes is 18. During the growth period of the normal spermatocytes the usual polarized leptotene, zygotene and pachytene stages occur. Nine tetrads are seen at diakinesis and at metaphase of the first maturation division.

The atypical spermatogenesis is shown schematically in figure 1. The atypical spermatocytes grow to several times as large as those of the normal line, the extra size being mostly in cytoplasmic volume. Slender leptotene-like threads may be identified, and these eventually condense to form compact chromosomes that assume a position at the nuclear periphery, like that characteristic of normal diakinesis. The chromosome threads are never polarized toward the idiozomal region; and there is nearly complete

absence of synapsis. The lack of synapsis is confirmed by the observation that at diakinesis there appear to be always a few more than 18 chromo-

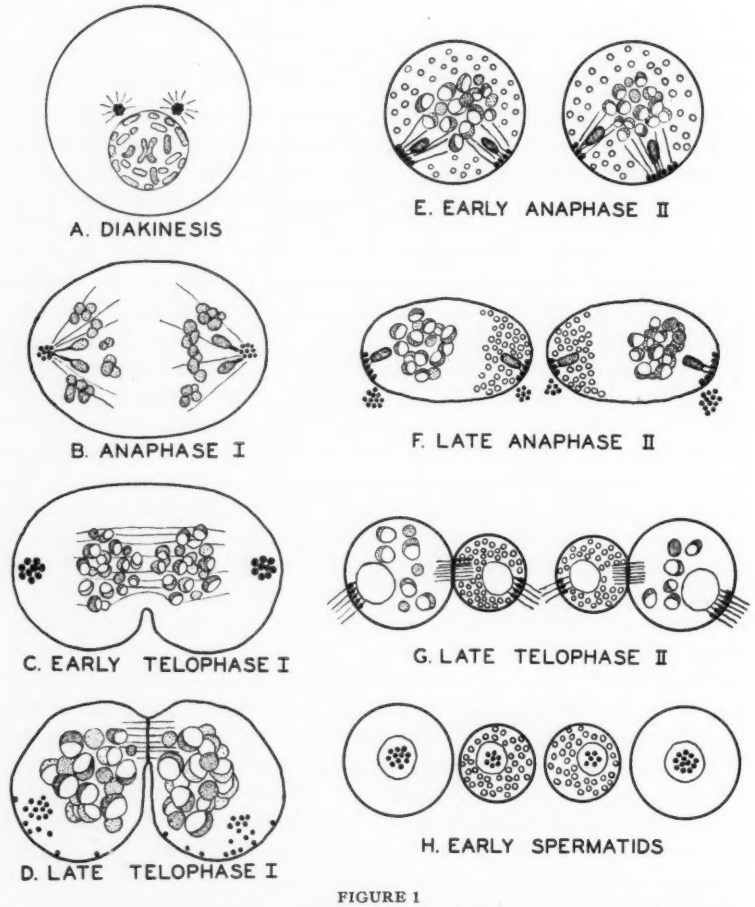


FIGURE 1

Schematized figures of successive stages in atypical spermatogenesis of Vivipara. A–G are similarly oriented. H shows spermatids in position for accurate counts of centrioles (see table 1). Centrioles are solid black, chromosomes stippled, chondriosomes (in E–H) are small circles.

somes; counts showing up to 23. With the exception of one probable tetrad these are simple rods like the chromosomes that are seen in spermatogonia (Fig. 1A). The presence of chromosomes in excess of the dip-

loid number is presumably due to precocious separation of some of the sister chromatids.

In the early atypical spermatocytes two centriole-like bodies may be distinguished in the idiozomal region. Both these increase in size, and near the end of the growth period they are large spheres, over one micron in diameter, and exactly equal in size. At diakinesis (Fig. 1A) the spheres appear mulberry-like, as Meves described them; they are near the nucleus; and each is at the center of a small aster. The spheres are some distance apart though never at opposite poles of the nucleus. As the nuclear membrane breaks down for the first division the chromosomes draw together into a compact ball, into which a number of lines (spindle fibres?) extend from each of the spherical groups of granules, which from now on will be referred to as centrioles. There is no metaphase plate stage. The two groups of centrioles move to opposite poles of the cell, and the chromosomes, most of them sticking together in several (4–8) clumps, accumulate near the centrioles (Fig. 1B). To each centriole group one or two separate chromosomes, isolated from the clumps that include most of the chromosomes, are connected by a definite, robust spindle fibre. Many fibres, however, extend from the centrioles in the general direction of the opposite pole without connecting with any chromosome or group of chromosomes. Just before cell constriction all of the chromosomes move away from the poles and the clumps break up into separate chromosomes. A preliminary small number of counts indicate that there are 34–36 chromosomes at this time. This is a close agreement with what is expected if one assumes that the separation of sister chromatids, apparently begun in diakinesis, has now been completed. The lower number agrees with expectation if the single tetrad noted at diakinesis has given rise not to four chromatids but to two dyads. In spite of these apparently rather disorderly mitotic phenomena each secondary spermatocyte receives very nearly the same number of chromosomes, and the cells formed by telophase constriction are about equal in size.

In late anaphase or early telophase (Fig. 1C) the individual centrioles grow, the groups of centrioles loosen up somewhat and it can be determined that the number in each cluster is almost certainly 9. During telophase each chromosome independently develops into a vesicle, with the chromatin concentrated in a thick cap on one side. These vesicles grow considerably during the interphase. In late telophase (Fig. 1D) each centriole group is located not far from the cell membrane, in a position asymmetrical with respect to both the spindle remnant and the plane of cell constriction, which has proceeded more rapidly from one side. At this time the number of centrioles is much greater than at earlier telophase; counts ranging from 14 to 16 are common. Since this is a stage before which each centriole of a typical normal spermatocyte would have divided it seems likely that in

these atypical cells likewise each of the 9 centrioles noted slightly earlier has now divided, and that the full number of centrioles in each daughter cell at late telophase is actually 18. The fact that the group of centrioles is breaking up and the individual centrioles are becoming dispersed throughout the cell probably accounts for failure to locate the full number. Most of the centrioles presently take up positions on the inner surface of the cell membrane, where they are seen during the second division, and until spermiogenesis is under way.

In the prophase of the second division the centrioles form two groups on the cell periphery, between 90 and 180 degrees apart. The large chromosomal vesicles decrease in size, and fibres may be seen penetrating from each centriole group into the cluster of these vesicles, which is in the approximate center of the cell. Next, two single chromosomes simultaneously leave the central mass, one migrating to each centriole group (Fig. 1E). These two chromosomes are not vesicles but are as fully condensed as those of the first division, and they appear perfectly normal. They are much alike in size and shape, and it is very plausible to regard them as mates. The other chromosomes do not condense beyond the small vesicle stage. Although they move slightly nearer to one pole they never approach it as closely as does the single chromosome that migrated earlier.

Just before cell constriction develops, small granular chondriosomes that were hitherto scattered throughout the cytoplasm become accumulated in the end of the cell opposite that where the chromosomal vesicles are located.

In many of these second maturation division figures one may observe that the number of centrioles in the two groups is markedly unequal; and in later stages the chondriosomes are massed near the smaller centriole group, while the chromosomes are closer to the larger (Fig. 1F). Cell constriction produces two unequal spermatids. The larger contains the greater number of centrioles and all but one of the chromosomes; while in the smaller cell are a single chromosome, the smaller centriole group, and nearly all the chondriosomes (Fig. 1G). On account of this last fortunate circumstance, throughout spermiogenesis it is easy to positively identify the smaller spermatids, not only by their smaller size but because of the heavier stain in the cytoplasm. Spermiogenesis differs in no important feature from that described by Meves in *V. vivipara*. The small nucleus in each spermatid is formed by the single chromosome that behaved normally in the second maturation division. All the abnormal chromosomes soon disappear. All the centrioles function as blepharoplasts, each producing a single axial filament.

The occurrence of the unequal second maturation division adequately accounts for the fact noted by Morita,[2] 1932, that in Japanese species of

Vivipara there are two sizes of atypical spermatozoa, which are always present in ratio of 1 : 1.

As the figures in table 1 indicate, the count of spermatid centrioles, which can be made with absolute certainty in cells that are favorably oriented (Fig. 1H), confirms the observation that the smaller spermatid is organized around the lesser number of centrioles, at least in 95% of the cases. The table is based on counts of 113 small cells and 128 large cells, being all those in which the number of centrioles could be determined in seven adjacent sections on one slide.

TABLE 1

NUMBER OF CENTRIOLES IN SMALL SPERMATIDS

Number of centrioles	*3*	*4*	*5*	*6*	*7*	*8*	*9*
% of cells	3.6	5.3	17.7	21.2	27.4	19.5	5.3

NUMBER OF CENTRIOLES IN LARGE SPERMATIDS

Number of centrioles	*15*	*14*	*13*	*12*	*11*	*10*	*9*
% of cells	7.0	6.3	12.5	23.4	28.9	19.5	2.4

If it be assumed that the number of centrioles apportioned to each two sister spermatids, i.e., the number in each secondary apermatocyte, is constant—which is probable from the previous history of the centrioles—then it is obvious that the data in table 1 agree best with the view that this constant secondary spermatocyte centriole number is 18. For convenience, in the table the complementary low and high numbers that add together in pairs to equal 18, have been placed in the same vertical column. The percentage of each should be equal to that of its complementary cell of the other size; and there is good agreement with this expectation, especially in the groups that include more cells where the numbers are large enough to be significant. The accuracy of this fit of the data to the number 18 will be emphasized if the reader tries fitting them to other possible numbers, e.g., 16 or 20, when it will be immediately apparent that one or more of the expected centriole counts is represented by a greatly deficient percentage of cells.

Analysis of the data in table 1 in a slightly different way is even more conducive to the same conclusion as to the centriole number in the secondary spermatocytes. If it were an invariable rule that a constant even number—an odd number is ruled out because of the occurrence of centriole division at the end of the first maturation division—of centrioles of the secondary spermatocytes were divided unequally between the large and the small spermatids, then the largest number of centrioles occurring in any of the small cells should be at least two less than the smallest number found

in any large cell. In the series of complementary numbers which added together in pairs equal the constant (secondary spermatocyte) number there should then be one number missing; and this missing number would be that which is just half of the constant total being divided (since by assumption there are no equal divisions of the group). In table 1 there is not actually any such gap, but there is a marked deficiency in the class containing 9 centrioles. The members of this class are so few that it is justifiable to regard them as resulting from equal allotment of the centrioles (a fact that is, of course, evident also from the circumstance that this is the only class represented by both large and small cells); and hence these cells are exceptions to the general rule of unequal apportionment of centrioles to the spermatids; and they may be disregarded, i.e., considered as amounting in effect to the gap in the series, as postulated above. Nine must then represent one-half of the total number of secondary spermatocyte centrioles, and we accordingly again conclude that that number is exactly 18.

Since the data in table 1 thus fully agree with the number that is expected if each of the 9 centrioles of the first spermatocyte group divided once in late telophase—which direct observation supports—it seems beyond reasonable doubt that the number of centrioles in each secondary spermatocyte is always exactly 18; that in four sister spermatids there are 36 centrioles, which in the course of spermatogenesis were derived by a single division of 18 centrioles that were earlier present as two equal groups of 9 each in the primary spermatocyte.

With the centriole and chromosome numbers that have been indicated above it is possible to make a highly suggestive comparison between the behavior of centrioles and that of chromosomes. In the typical normal first spermatocyte there are *two* centrioles. These normal centrioles divide once in the first maturation to give 4, one of which enters each spermatid and there functions as a blepharoplast. Comparing the atypical first spermatocyte with the typical normal one, in the former there are *18 centrioles*, in other words, there are *16 extra centrioles*. The haploid number of normal chromosomes is 9, the diploid 18. In the normal late spermatocyte there are 9 tetrads—equivalent to *18 normal dyads*, or to 36 normal chromatids. In the atypical line, by contrast—as indicated by the very specific chromosome behavior in the atypical second maturation division, whereby each spermatid receives one chromosome (single chromatid)—there are but 4 of the original 36 chromatids that function normally in the maturation divisions and remain intact in the spermatid. This idea of four normal chromatids is further supported by the observation of a single tetrad at diakinesis, and possibly also by the events at anaphase of division one. *This leaves then 32 chromatids, originally in the form of 16 dyad-chromosomes, that are abnormal.*

This coincidence of the occurrence of *16 abnormal dyads* and the simul-

taneous presence of *exactly 16 extra centrioles* is too striking to be lightly dismissed, especially since this is only one of a large number of cases where it has been demonstrated that increase in centriole number is accompanied by this same sort of abnormal chromosome behavior. Expressed very simply it looks as if 16 chromosomes have each lost something, while the cytoplasm has at the same time gained 16 units, the extra centrioles. That the centriole is intimately involved with the chromosomes has long been evident from the fact that centrioles grow and divide only immediately after they have been in close relation with the chromosome complement, i.e., located at the poles of the spindle—a generalization about centrioles to which these cases of atypical gasteropod spermatogenesis have hitherto seemed to offer the only real objection. Schrader,[3] 1936, and Darlington,[4] 1937, have recently called attention to numerous similarities between the centrioles and the kinetochore, or spindle fibre attachment point, of the chromosome. The failure to assume normal relation with the spindle is a feature of these abnormal chromosomes in the atypical spermatogenesis of Vivipara and of other prosobranch snails, which is precisely the sort of behavior that would be expected if there were a marked disturbance (complete absence?) of the normal kinetochore function. The increase in number of centrioles and the abnormal chromosome behavior in the atypical spermatogenesis of Vivipara are both adequately explained if one assumes that the kinetochore masses of 16 dyad-chromosomes have been transferred from nucleus to cytoplasm, where each has given rise to a perfectly normal centriole that is in no way distinguishable from the two already present there and always present in each typical normal cell.

[1] F. Meves, *Arch. f. mikr. Anat.*, **61**, 1–84(1902).
[2] J. Morita, *Fol. anat. Jap.*, **10**, 35–51(1932).
[3] F. Schrader, *Biol. Bull.*, **70**, 484–498(1936).
[4] C. D. Darlington, *Recent Advances in Cytology*, Blakiston, Philadelphia (1937).

AN ERGODIC THEOREM FOR n-PARAMETER GROUPS

BY NELSON DUNFORD

DEPARTMENT OF MATHEMATICS, YALE UNIVERSITY

Communicated March 9, 1939

Let G stand for either the whole of Euclidean n-space or those points $\alpha = (a_1, \ldots, a_n)$ for which $a_i \geqslant 0$. Let T_α $(\alpha \in G)$ be a linear transformation in a Banach space $\mathfrak{B}$, with $||T_\alpha|| \leq C$, $T_{\alpha+\beta} = T_\alpha T_\beta$, $(\alpha, \beta \in G)$. We suppose that for each x in $\mathfrak{B}$ the set of points $T_\alpha x (\alpha \in G)$ is a separable subset of $\mathfrak{B}$ and that for each $\bar{x} \in \bar{\mathfrak{B}}$ (the space conjugate to $\mathfrak{B}$) and x in $\mathfrak{B}$ the numerical

function $\bar{x}T_\alpha x$ is measurable in the sense of Lebesgue. Under these assumptions the $\mathfrak{B}$ valued function $T_\alpha x$ is (for x fixed) bounded on every cube I and may be approached almost everywhere on I by step functions. The function $T_\alpha x$ is thus absolutely integrable and measurable over every cube I. Let $I_r \subset G$ be any cube of side $r > 0$ and let $\mathfrak{M}$ consist of those points $x \in \mathfrak{B}$ for which the set $y_r = (1/r^n)\int_{I_r} T_\alpha x d\alpha$ is weakly compact with respect to $r \to \infty$. Then $\mathfrak{M}$ is a closed linear manifold in $\mathfrak{B}$ and there is a projection U of $\mathfrak{M}$ into part of itself with

$$T_\alpha Ux = Ux, \quad x \in \mathfrak{M},$$
$$Ux = \lim_{r \to \infty} (1/r^n)\int_{I_r} T_\alpha x d\alpha, \quad x \in \mathfrak{M}.$$

The set $U\mathfrak{M}$ is the closed linear manifold consisting of those x in $\mathfrak{B}$ for which $T_\alpha x = x$, $(\alpha \in G)$. Furthermore we have $\mathfrak{M} = \mathfrak{B}$ in case $\mathfrak{B}$ satisfies one of the following conditions: (a) its unit sphere is weakly compact, (b) it is reflexive, (c) it has an equivalent uniformly convex norm.

The space $L(E, m)$ of real functions which are measurable with respect to a Borel field $\mathfrak{A}$ of subsets of E and which are summable on E with respect to a finite non-negative completely additive set function m or $\mathfrak{A}$ is a Banach space which fails to satisfy any of the above conditions (a), (b), (c). In this situation we have also $\mathfrak{M} = \mathfrak{B} = L(E, m)$ in case the group T_α is generated by a measure preserving group of point transformations $S(\alpha, P)$ having the property that the function $x(S(\alpha, P))$ (for x in $L(E, m)$) is measurable in the product space $G \times E$. In the classical case, i.e., the one just mentioned, the properties of $S(\alpha, P)$ imply all the assumptions made on the abstract group T_α.

The result $\mathfrak{M} = L(E, m)$ for the classical case follows from the general theorem by means of a lemma which states that a set F in $L(E, m)$ is weakly compact if and only if $\int_E |f(P)|\, dm$ is bounded uniformly for f in F and $\lim_{m(\sigma)=0} \int f(P)dm = 0$ uniformly for f in F.

The method of proof used in the general theorem is an extension of one used by F. Riesz[1] for the discrete case in one dimension. We first show that any weak limit y of any subsequence $y_{r_p}(r_p \to \infty)$ of y_r is invariant under the group. It is then shown that y is actually the strong $\lim_{r \to \infty} (1/r^n)\int_{I_r} T_\alpha u d\alpha$ where u is the average value of $T_\alpha x$ over any cube. The desired result then follows from the differentiability properties of the abstract integral $\int_I T_\alpha x d\alpha$. The details of calculation will be published elsewhere.

[1] Riesz, F., *Jour. Lond. Math. Soc.*, **13**, 274–278 (1938).

ON ANALOGUES OF THE BERNOULLI AND ALLIED NUMBERS

BY H. S. VANDIVER

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated February 21, 1939

In a former article[1] in these PROCEEDINGS I gave a result (Theorem 1) generalizing the von-Staudt-Clausen theorem, for the Bernoulli numbers themselves, so as to apply to numbers of the form $(mb + k)^n$ where m and k are integers, $m \neq 0$, and this is a symbolic power, i.e., we expand by the binomial theorem and then set $b^i = b_i$, the b's being defined by the recursion formula

$$(b + 1)^s = b_s; \quad s > 1. \tag{1}$$

with

$$(mb + k)^0 = 1; \quad 0^0 = 1.$$

Also if we consider

$$(kb' + jb'')^a; \quad a > 0;$$

where k and j are positive integers and this expression is expanded in full and b_i is substituted for $(b')^i$ and also for $(b'')^i$, I stated in the previous paper that this expression can be reduced to a linear combination of numbers of the type $(kb + l)^n$ and $(jb + h)^n$; $n = a, a - 1$, and a proof was indicated briefly. Since then a number of extensions of those ideas have been found, and we shall state some of them here. Full proofs I hope to give elsewhere. In most of this work a fundamental rôle is played by the identity[2]

$$\frac{x^j - y^k}{(x - 1)(y - 1)} = \frac{\sum_{n=0}^{j-1} x^n y^{k-a_n}}{y - 1} - \frac{\sum_{l=0}^{k-1} y^l x^{j-b_l}}{x - 1}$$

where x and y are transcendental adjunctions to the rational field (indeterminates), $a_n = [nk/j]$, $b_l = [lj/k]$; $[u]$ is the greatest integer in u, j and k being positive integers. This relation is manipulated in various ways to obtain a variety of results. We have

THEOREM 1. *If h, j and k are integers $k > 0, j > 0$, c_i represents the least positive or zero residue of ki, modulo j; d_l denotes the least positive or zero residue of lj, modulo k; $n > 0$, then*

$$jk(kb' + jb'' + h)^n = (1 - n)jk(b + h)^n + hjkn(b + h)^{n-1}$$
$$+ k^2n \sum_{i=0}^{j-1} i(jb + c_i + h)^{n-1} + j^2n \sum_{l=0}^{d-1} l(kb + d_l + h)^{n-1}.$$

There is an analogue of this theorem for $kj < 0$. From these results and the properties of the numbers $(mb + k)^n$ we obtain

THEOREM 2. *If h, j and k are integers, n even and > 0, then*

$$jk(kb' + jb'' + h)^n = I + (1 - n)jk \sum_{i=1}^{s} \frac{1}{p_i}$$

where the p's are the distinct primes such that $n \equiv 0 \pmod{p_i - 1}$; I being some integer.

We note that this is *not* a generalization of the Theorem 1 of my previous paper since the formula does not hold for either k or $j = 0$. However, it is an analogue of that theorem. There is also an analogue of Theorem 1, above, for the Bernoulli number of the rth order,

$$(m_r b^{(r)} + m_{r-1} b^{(r-1)} + \ldots + m_1 b' + m_0)^n$$

$$m_i \neq 0,\ i = 1, 2, \ldots, r.$$

It may be found by induction, but the explicit formula is complicated. *In general, speaking roughly, our investigations indicate that if we have any relation involving the Bernoulli numbers and the binomial theorem, there is an analogous result involving the multinomial theorem.*

Letting

$$(mb + k)^n = b_n(m, k)$$

then a recursion formula for $b_n(m, k)$ is

$$(b(m, k) + m)^{n+1} - b_{n+1}(m, k) = k^n m(n + 1)$$

which reduces to (1) for $k = 0$, $m = 1$, and holds for $n = 0$ since $0^0 = 1$ by convention. Here m and k can be any elements of a ring which includes the rational field. If we take the recursion formula

$$(H + 1)^n = xH_n \tag{2}$$

where x is an indeterminate, and on expansion we substitute H_i for H^i, then we find that

$$H_n = \frac{R_n(x)}{(x - 1)^n}$$

where $R_n(x)$ is a polynomial with integral coefficients. Then

$$R_1(x), \qquad R_2(x), \ldots .$$

are called the *Euler polynomials*. They may be generalized by considering the functions

$$H_i(m, k)$$

obtained from the recursion formula

$$(H + m)^n = xH_n + (-k)^n(1 - x)$$

which reduces to (2) for $k = 0$, $m = 1$. Here again m and k can belong to any ring containing the rational field, in particular they may be indeterminates. We find that

$$H_n(m, k) = \frac{R_n(m, k)}{(1 - x)^n}$$

where $R_n(m, k)$ is a polynomial in m, k and x with integral coefficients. We also have

$$H_n(x, m, k) = (-1)^n(1 - x)\left[\frac{d^n}{dv^n}\frac{e^{v(k+m)}}{1 - xe^{vm}}\right]_{v=0} \tag{3}$$

and

$$f_n(x, m, k) - k^n(1 - x^p) \equiv (-1)^n x \frac{1 - x^p}{1 - x} H_n(x, m, k) \tag{4}$$

modulo p, a prime, where

$$f_a(x, m, k) = k^a + (k + m)^a x + (k + 2m)^a x^2 + \ldots + (k + (p - 1)m)^a x^{p-1}.$$

The relations (3) and (4) reduce to known formulas for $m = 1$, $k = 0$.

Set

$$f_a(x, 1, 0) = f_a(x);$$

then we find, if Σ'/ρ indicates summation over all the distinct mth root of unity other than unity and Σ'/ζ indicates summation over all the distinct Σ'/kth roots of unity other than unity,

$$\frac{(kb + mb')^a}{a(a - 1)} + \frac{b_{a-1}}{2(a - 1)}(k + m) + \frac{b_a}{a} \equiv k\sum_{\rho}{}' \frac{(-1)^a f_{a-2}(\rho)}{(\rho^k - 1)(1 - \rho^p)} + m\sum_{\zeta}{}' \frac{(-1)^a f_{a-2}(\zeta)}{(\zeta^m - 1)(1 - \zeta^p)} \tag{5}$$

modulo p, where $m > 1$, $k > 1$; $2 < a < p$ with m, k and p prime each to each.

From this we may obtain an equation instead of a congruence by employing the known relation, with $x = \rho$ and ζ in turn,

$$f_n(x) \equiv x(1 - x)^{p-n-1} R_n(x) (\text{mod } p) \tag{5a}$$

and noting that the resulting terms after reduction are independent of p.

The left-hand member of (5) has properties analogous to that of the generalized tangent function

$$\frac{(mb + k)^n - b_n}{n} = t_n(m, k)$$

mentioned in the previous paper. In analogy to (5) we have, if $m > 1$, $(m, p) = 1$,

$$t_n(m, k) \equiv \sum_{\rho}{}' \frac{(-1)^{n+1}\rho^k f_{n-1}(\rho)}{1 - \rho^p} + \sum_{l=0}^{k-1}\sum_{\rho} l^{n-1}\rho^{l-k} \qquad (6)$$

modulo p, where Σ/ρ indicates summation over all the mth roots of unity. This is due to Frobenius[3] for $k < m$ and in that case the last term on the right vanishes. The relation (6) may also be given in the form, if $k > m$, $(m, p) = 1$, $m > 1$,

$$t_n(m, k) \equiv \sum_{\rho}{}' \frac{(-1)^{n+1}\rho^k f_{n-1}(\rho)}{1 - \rho^p} + m\sum_{s=1}^{\left[\frac{k}{m}\right]} (k - sm)^{n-1}, \qquad (7)$$

modulo p. This is a companion formula to the well-known relation

$$\frac{(b + k)^n - b_n}{n} = \sum_{s=0}^{k-1} s^{n-1},$$

but the latter is not obtainable from (7) since (7) does not hold for $m = 1$. As before (6) and (7) yield equalities involving $R_a(\rho)$ by employing (5a).

As a by-product of (5) we obtain the congruence if $a > 1$, a even, $(m, k) = 1$, $(m, p) = 1$,

$$\frac{(1 - m^a)b_a}{2a} \equiv \sum_{\rho}{}' \frac{f_{a-1}(\rho)}{(1 - \rho^p)(1 - \rho^k)} \pmod{p} \qquad (8)$$

but there is no corresponding relation for a odd.

There is a class of congruences involving Bernoulli numbers known as Kummer's congruences,[4] as follows:

$$h^n(h^{p-1} - 1)^j \equiv 0 \pmod{p^j} \qquad (9)$$

where $(n - 1) \geqq j$; $n \not\equiv 0 \pmod{p - 1}$, the left-hand member being expanded in full by the binomial theorem and b_t/t is substituted in the result in place of h_t. Inspection shows that (9) is not true for $n \equiv 0 \pmod{p - 1}$ and as far as I know, a complementary theorem to cover this case has not been published. We have found one such, and it is as follows:

$$b^{a(p-2)}(b^{p-1} - 1)^j \equiv 0 (\text{mod } p^{j-1}) \qquad (10)$$

where $a > 0, j > 0, a + j < p - 1$.

[1] These PROCEEDINGS, **23**, 556 (1937).
[2] Vandiver, *Ann. Math.*, (2), 29, 171 (1928).
[3] Berlin Sitsungsberichte, *Math. Phys. Kl.*, 653 (1914).
[4] For a proof, with generalizations, cf. Vandiver, *Bull. Amer. Math. Soc.*, 422 (1937).

GENERAL SELF-ADJOINT BOUNDARY CONDITIONS FOR CERTAIN PARTIAL DIFFERENTIAL OPERATORS

BY J. W. CALKIN

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE

Communicated March 7, 1939

In a previous note we have described briefly an abstract theory of self-adjoint boundary conditions which can be applied to the study of a wide class of differential operators.[1] As the result of investigations carried on in part prior to and in part in conjunction with the development of that theory, we have now accumulated a considerable amount of information concerning its application to partial differential operators of the form

$$L \equiv -\frac{\partial}{\partial x} p(x, y) \frac{\partial}{\partial x} - \frac{\partial}{\partial y} p(x, y) \frac{\partial}{\partial y} + q(x, y) \cdot . \qquad (1)$$

However, it appears desirable to postpone the publication of this material in detail until various further investigations can be carried out; in consequence, we present here without detailed proofs, some of the results so far obtained.[2] As in (A), we use the notation and terminology of Stone,[3] except for minor modifications.

The first two theorems which we state provide information of a preliminary sort, most of which is necessary before we can bring the general theory to bear.

THEOREM 1. *Let E be an open set in the (x, y)-plane, let q be a real-valued Lebesgue-measurable function defined on E and bounded on every closed set interior to E, and let p be a real-valued function satisfying the condition (i) below and the additional restriction that $1/p$, p_{xx} and p_{yy} be bounded on every bounded closed set interior to E. Let $\mathfrak{D}^*$ be the set of all functions f in the complex Hilbert space $\mathfrak{L}_2(E)$ which have the following properties:*

(i) interior to E, on almost all lines $y = a$ which intersect E, f and f_x are absolutely continuous in x; and on almost all lines $x = a$ which intersect E, f and f_y are absolutely continuous in y;

(*ii*) *the integrals*

$$\int_S |f_x|^2\,dS, \quad \int_S |f_y|^2\,dS, \quad \int_S |f_{xx}|^2\,dS, \quad \int_S |f_{yy}|^2\,dS$$

exist on every bounded closed set S interior to E;

(*iii*) *the integral* $\int_E |L(f)|^2 dE$ *exists, where L is given by* (1).

Let T be the transformation in $\mathfrak{L}_2(E)$ *with domain* $\mathfrak{D}^*$ *which takes each element f in its domain into L(f). Then T is a closed linear transformation in* $\mathfrak{L}_2(E)$ *and has an adjoint H which is symmetric;* $H \equiv T^*$, $T \equiv H^*$, $H \subseteq H^*$. *The deficiency-index of H is* (n, n) *and all of the cases,* $n = 0, 1, 2, \ldots, \aleph_0$, *occur.*

If S is an arbitrary maximal symmetric extension of H, or the adjoint of such an extension, then the resolvent of S is an integral operator with kernel $G(P, Q; \lambda)$ *of Carleman type. For each value of* λ, $G(P, Q; \lambda)$ *is continuous on* $E \times E$ *except on the set* $P = Q$ *where it has a logarithmic singularity; for fixed Q,* $G(P, Q; \lambda)$, *considered as a function of the coördinates* (x, y) *of P, satisfies the condition* (*i*) *above and the equation* $L(G) - \lambda G = 0$. *The kernel of the transformation* $(S^* - \bar{\lambda}I)^{-1}$ *is* $\overline{G(Q, P; \lambda)}$.

We point out that a function f of $\mathfrak{L}_2(E)$ is considered to have the property (*i*) if it is equal almost everywhere on E to a function which has that property in the strict sense; the derivatives of f are of course to be calculated by operating on the latter function. For what follows it is important also for us to note that a function f which has the properties (*i*)–(*iii*) is equal almost everywhere on E to a continuous function with the same properties.

For convenience we now introduce

DEFINITION 1. *An open set E in the* (x, y)*-plane is said to be of Class* I *if it is simply-connected, has a rectifiable boundary C, and if a function* $f(z) = f(x + iy)$ *which maps E conformally on the interior of the unit circle has the property that* $|f'(z)|$ *and its reciprocal remain bounded as z approaches any point of C.*

THEOREM 2. *Let the set E of Theorem* 1 *be of Class* I; *let the functions q,* $1/p$, p_{xx}, p_{yy} *be uniformly bounded on E; and let p be continuous on* $E + C$. *Let* $\mathfrak{D}_1^*$ *be the set of all elements f of* $\mathfrak{D}^*$ *which have the following properties: f has continuous boundary values on C; at almost all points* (ζ, η) *of C,*

$$f_x(\zeta, \eta) = \lim_{x \to \zeta} f_x(x, \eta) \text{ and } f_y(\zeta, \eta) = \lim_{y \to \eta} f_y(\zeta, \eta)$$

exist and are measurable functions on C; if $x = x(s)$, $y = y(s)$ *is the parametric representation of C in terms of arc-length,*

$$\partial f/\partial n = -f_x y'(s) + f_y x'(s)$$

is of integrable absolute value square on C. Then for every pair of elements f and g of $\mathfrak{D}_1^*$, *the formula*

$$\int_E [fL(\bar{g}) - \bar{g}L(f)]dE - \int_C p\left(f\frac{\partial\bar{g}}{\partial n} - \bar{g}\frac{\partial f}{\partial n}\right)ds = 0 \quad (2)$$

is valid, the integral over C existing in the sense of Lebesgue.

Let T_1 be the contraction of T with domain $\mathfrak{D}_1^*$. Then $\tilde{T}_1 \equiv T \equiv H^*$ and $T_1 \supset H$. The domain $\mathfrak{D}$ of H consists of these and only these elements g of $\mathfrak{D}_1^*$ such that $g = \partial g/\partial n = 0$ almost everywhere on C. H has the deficiency-index $(\aleph_0, \aleph_0)$.

At two points the formal statement of Theorem 2 requires elucidation. First, the requirement that p be continuous on $E + C$ is not at all restrictive, since the other conditions on p imply that it is at least equivalent to a function continuous on $E + C$. Second, in discussing the behavior at C of an element f of $\mathfrak{D}_1^*$ we have reference to the continuous representative of f.

We now state two closely related theorems, the second of which reveals the formula (2) above as a realization of the formula (1) of (A).

In both of these theorems, and in all subsequent ones, it is assumed that the hypotheses of Theorem 2 are satisfied.

THEOREM 3. *Let $\mathfrak{T}$ denote the set of all elements $\{f(x, y), h(s)\}$ of $\mathfrak{L}_2(E) \oplus \mathfrak{L}_2(C)$ such that f is in $\mathfrak{D}_1^*$ and $h(s)$ is the value of f on C. Let D be the transformation in $\mathfrak{L}_2(E) \oplus \mathfrak{L}_2(C)$ with domain $\mathfrak{T}$ which takes each element $\{f(x, y), f(s)\}$ of its domain into $\{L(f), -p\partial f/\partial n\}$. Then D is a not-negative definite self-adjoint transformation in $\mathfrak{L}_2(E) \oplus \mathfrak{L}_2(C)$. The resolvent of D is totally continuous—is, in fact, of finite norm.*[4]

THEOREM 4. *Let A be the transformation with domain the graph of T_1, which takes each element $\{f, H^*f\}$ of its domain into the element $\{f, -p\partial f/\partial n\}$ of $\mathfrak{L}_2(C) \oplus \mathfrak{L}_2(C)$. Then A is an unbounded reduction operator for H^*, with range-space $\mathfrak{L}_2(C) \oplus \mathfrak{L}_2(C)$. The unitary transformation W associated with A is the transformation in $\mathfrak{L}_2(C) \oplus \mathfrak{L}_2(C)$ which takes $\{h(s), k(s)\}$ into $\{k(s), -h(s)\}$.*

Theorem 4 follows at once from Theorem 3 and the fact that $\tilde{T}_1 \equiv H^*$.

We point out here that in proving Theorem 3, we have made use of known facts concerning the solution of the Dirichlet and Neumann problems for the set E. In consequence, certain subsequent results, which can be specialized into assertions concerning those problems, cannot be interpreted as yielding new proofs of known results in the theory of the logarithmic potential. We have not, however, employed the results of any other previous work on boundary value problems.

Because of the special character of A, revealed in the fact that its graph is also the graph of the self-adjoint operator D (when we regard the operation $\oplus$ as commutative), the symmetric boundary conditions associated with H^* by A can be formulated in a somewhat simpler form than is immediately indicated by the general theory of (B). Thus, we have

THEOREM 5. *Let V be an arbitrary isometric transformation in $\mathfrak{L}_2(C)$.*

Then the set of all elements f of $\mathfrak{D}_1^*$ such that, for some h in the domain of V,

$$f(s) = h - Vh, \quad -p\frac{\partial f}{\partial n} = i(h + Vh), \tag{3}$$

constitutes the domain of a symmetric extension $H(V)$ of H. If S is an arbitrary symmetric extension of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$, there exists at least one isometric transformation V in $\mathfrak{L}_2(C)$ such that $S \equiv H(V)$.

Theorem 5 is readily proved on the basis of the abstract theory cited, or can be verified directly by reference to the formula (2) above.

It is to be noted that we may have $H(V_1) \equiv H(V_2)$, $V_1 \neq V_2$, even when V_1 and V_2 are both closed.

It is now possible to interpret the conditions of (A) Theorem 5, or of (B) Theorem 4.8, as conditions for the maximality of the transformation $H(V)$ of Theorem 5 above. A further simplification, however, results from the fact that D has a totally continuous resolvent. When we take account of this fact in the light of (*B*) Chapter III, §4 and Chapter IV, §3, we find that, in the terminology introduced there, A is of Type I with characteristic index (0, 0). In consequence, A exhibits in certain respects a peculiar resemblance to reduction operators whose range-spaces are unitary; or, to be more precise, we have

THEOREM 6. *Every maximal symmetric extension of S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ is self-adjoint. The class of all such extensions S has the cardinal number of the continuum.*

In view of Theorem 6 and certain other special properties of the operator A of Theorem 4, the application of (A) Theorem 5 now yields

THEOREM 7. *Let $\mathfrak{F}$ be the set of all elements h of $\mathfrak{L}_2(C)$ such that the equations $f(s) = h$, $-p\partial f/\partial n = ih$ have a solution f in $\mathfrak{D}_1^*$. Let K be the transformation in $\mathfrak{L}_2(C)$ which satisfies the equation $X\{0, h\} = \{g, Kh\}$ for every h in $\mathfrak{L}_2(C)$, X being the Cayley transform of D. Then K is bounded linear. For the transformation $H(V)$ of Theorem 5 above to be self-adjoint either of the conditions*

$$\mathfrak{R}(V - K) \supseteq \mathfrak{F}, \quad \mathfrak{R}(V^{-1} - K^*) \supseteq \mathfrak{F}, \tag{4}$$

is necessary and sufficient.

In so far as we have been able to determine, the set $\mathfrak{F}$ of Theorem 7 depends on the boundary C, but not on the operator L under the restrictions already imposed on the latter. It can be shown that $\mathfrak{F}$ is linear and everywhere dense in $\mathfrak{L}_2(C)$ and that every element h of $\mathfrak{F}$ is absolutely continuous and has its first derivative in $\mathfrak{L}_2(C)$. Thus the conditions (4) yield somewhat simpler sufficient conditions for self-adjointness, which are not, however, necessary.

THEOREM 8. *In particular, the conditions (4) of Theorem 7 are satisfied if $V \equiv I$, or if V is unitary and $(I - V)^{-1}$ exists and is bounded. In the*

first case, the boundary condition (3) *is equivalent to the requirement* $f(s) = 0$; *in the second, it is equivalent to the condition* $p\partial f/\partial n + Gf(s) = 0$, *where* G *is the self-adjoint transformation* $i(I + V)(I - V)^{-1}$ *in* $\mathfrak{L}_2(C)$.

The assertion concerning the case $V \equiv I$ is based on a special analysis of the boundary condition in question. That the boundary condition $p\partial f/\partial n + Gf(s) = 0$, where G is bounded self-adjoint, defines a self-adjoint extension of H follows from (B) Theorem 4.18.

In passing, we may note that the boundary conditions of Theorem 8 are non-degenerate in the sense of (A) Definition 2. This is not true of all boundary conditions satisfying the conditions (4), although we can associate with each self-adjoint extension S of H such that $\mathfrak{B}(S) \subset \mathfrak{D}(A)$ a non-degenerate boundary condition such that the definitive transformation V is closed.

THEOREM 9. *Let* V *satisfy the conditions* (4). *Then the kernel* $G(P, Q; \lambda)$ *of the resolvent of* $H(V)$ *is of Hilbert-Schmidt type. If* Q *is a fixed point interior to* E, $G(P, Q; \lambda)$ *considered as a function of the coördinates of* P *has continuous boundary values on* C, *possesses a normal derivative there in the sense of Theorem* 2, *and satisfies the boundary condition* (3) *defining* $H(V)$. *The spectrum of* $H(V)$ *consists solely of characteristic values, each with finite multiplicity. If* $\{\lambda_n\}$ *is the sequence of characteristic values of* $H(V)$, *excluding* $\lambda = 0$ *when it occurs, and if* α_n *is the multiplicity of* λ_n, *then*

$$\sum_{n=1}^{\infty} \alpha_n / |\lambda_n|^2 < \infty.$$

We omit any discussion of the proof of Theorem 9, except to observe that the fact that $G(P, Q; \lambda)$ is of Hilbert-Schmidt type is a consequence of the fact that the resolvent of D has finite norm.

All of the results stated above are of a positive nature. Further ones—concerning conditions for $H(V)$ to have a maximal symmetric closure—can be obtained by recourse to (B), Theorems 4.9–4.12. Other results which are best described as negative and which reveal Theorem 7 and those unstated theorems just suggested as essentially the "best possible," can be obtained by reference to the discussion in (B) of the pathological aspects of the behavior of unbounded reduction operators. Of results of this sort, we mention here only two. First, there exist self-adjoint extensions S of H such that $\mathfrak{B}(S) \cdot \mathfrak{D}(A) = \mathfrak{B}(H)$. Second, there exist unitary transformations V in $\mathfrak{L}_2(C)$ such that the equations (3) have a solution f in $\mathfrak{D}_1^*$ for every h in $\mathfrak{L}_2(C)$ and such that $H(V)$ is closed and has the deficiency-index (n, n), $n = 1, 2, \ldots, \aleph_0$.

We conclude with a few remarks concerning homogeneous boundary value problems. First, the differential system $L(f) - \lambda f = 0$, $f(s) = h$, has a unique solution f in $\mathfrak{D}^*$ for every h in $\mathfrak{L}_2(C)$, except when λ is a characteristic value of $H(I)$. The solution f is in $\mathfrak{D}_1^*$ if and only if h is

absolutely continuous and has a derivative in $\mathfrak{L}_2(C)$. Second, if G is a bounded self-adjoint transformation in $\mathfrak{L}_2(C)$, the system $L(f) - \lambda f = 0$, $p\partial g/\partial n + Gg = h$, has a unique solution f in $\mathfrak{D}_1{}^*$ for every h in $\mathfrak{L}_2(C)$, except when λ is in the spectrum of $H(V)$, $V \equiv (G - iI)(G + iI)^{-1}$.

[1] These PROCEEDINGS, **24**, 38–42 (1938). A detailed account of this theory will appear in *Trans. Amer. Math. Soc.* under the title *Abstract Symmetric Boundary Conditions*. In the sequel we refer to the first of these papers as (A), to the second as (B).

[2] Some of the results appeared in the writer's doctoral thesis, Harvard, 1937.

[3] M. H. Stone, *Linear Transformations in Hilbert Space*, New York, 1932.

[4] Compare L. Lichtenstein, *Math. Zeit.*, **3**, 127–160, esp. 151–160 (1919).

ON THE GAUSSIAN LAW OF ERRORS IN THE THEORY OF ADDITIVE FUNCTIONS

BY P. ERDÖS AND M. KAC

THE INSTITUTE FOR ADVANCED STUDY AND THE JOHNS HOPKINS UNIVERSITY

Communicated March 13, 1939

In the present note we state without proofs some results concerning additive functions, the proofs of which depend partially on statistical methods. A function $f(m)$ is called additive if for $(m_1, m_2) = 1$ one has $f(m_1 \cdot m_2) = f(m_1) + f(m_2)$. We assume furthermore that $f(p^\alpha) = f(p)$ and $|f(p)| \leqslant 1$ for every prime p. None of these assumptions is essential but they simplify the statement of Theorem A.[1]

THEOREM A. *Let $f(p)$ be such that*

$$F(n) = \sum_{p < n} \frac{f^2(p)}{p}$$

diverges. Then the density of integers for which

$$f(m) < \sum_{p < m} \frac{f(p)}{p} + \omega \sqrt{2F(\ )}$$

is equal to $\pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2)dy$ *for any real* ω.

The proof depends on the following two lemmas.

LEMMA 1. *Let p_k be the kth prime and let*

$$f_k(m) = \sum_{\substack{p/m \\ p \leqslant p_k}} f(p).$$

Further let $\delta(k)$ be the density of the integers which satisfy the inequality

$$f_k(m) < \sum_{p \leqq p_k} \frac{f(p)}{p} + \omega \sqrt{2 \sum_{p \leqq p_k} \frac{f^2(p)}{p}} . \qquad (1)$$

Then

$$\lim_{k \to \infty} \delta(k) = \pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2) dy.$$

The proof depends on the use of Fourier transforms.

LEMMA 2. *Let $n = k^{\varphi(k)}$, where $\varphi(k)$ tends to ∞ as k tends to ∞ arbitrarily slowly.*

Let $\psi(k, n)$ be the number of integers $\leqq n$ satisfying (1), *and let $\delta(k, n) = \psi(k, n)/n$.*

Then

$$\lim_{k \to \infty} \delta(k, n) = \lim_{k \to \infty} \delta(k) = \pi^{-1/2} \int_{-\infty}^{\omega} \exp(-y^2) dy.$$

In order to deduce this lemma from the previous one we need Brun's method.

The proof of Theorem A now follows easily by elementary methods.[2]

From Theorem A, putting $\omega = 0$, one immediately deduces the following result:

The density of the integers which satisfy the inequality

$$f(m) < \sum_{p \leqq m} \frac{f(p)}{p}$$

is equal to $1/2$.

In the special case $f(m) = \nu(m)$ ($\nu(m)$ denotes the number of different prime divisors of m) this was proved by Erdös.[3]

[1] It suffices to assume that $\sum_{|f(p)|>1} \frac{1}{p}$ converges.

[2] Compare P. Erdös, "On a Problem of Chowla and Some Related Problems," *Proc. Camb. Phil. Soc.*, **32**, 530–540 (1936).

[3] "Note on the Number of Prime Divisors of Integers," *Jour. Lond. Math. Soc.*, **11**, 308–314 (1936).

ON THE CONNECTIVITY RING OF A LATTICE

BY J. W. ALEXANDER

INSTITUTE FOR ADVANCED STUDY

Communicated March 9, 1939

The topological theory of abstract spaces takes on a simpler and more general form if we think of a space not as a collection of points but as a lattice (which need not have atomic elements). Pending a more detailed treatment of the general theory which will shortly be given elsewhere, we wish merely to explain how the connectivity ring of a lattice may be defined.

Let L be an arbitrary distributive lattice with zero element (but, in general, without unit element). Then with the lattice L there is associated an *enlargement* L^* of L consisting of all transformations x^* of L into itself, such that each x^* has the following four properties:

(1) The transformation is a homomorphism. That is, if it carries x_1 into x_1' and x_2 into x_2' it carries $x_1 \smile x_2$ into $x_1' \smile x_2'$ and $x_1 \frown x_2$ into $x_1' \frown x_2'$.

(2) It is a projection. That is, if it carries x into x' it carries x' into itself.

(3) It carries each element x of L into an element x' such that $x' \leqq x$.

(4) If it leaves invariant an element x it also leaves invariant every element x_1 such that $x_1 \leqq x$.

Given any two transformations x_1^* and x_2^* of L^* carrying the element x of L into the elements x_1' and x_2', respectively, we define their *union* $x_1^* \smile x_2^*$ as the transformation carrying x into $x_1' \smile x_2'$ and their *intersection* $x_1^* \frown x_2^*$ as the transformation carrying x into $x_1' \frown x_2'$. Thus, the set L^* is also a lattice. Moreover, it is easy to see that L^* is distributive and that it possesses both a zero element (transforming every x into 0) and a unit element (transforming every x into itself). Finally, we can identify the lattice L with a sub-lattice of L^* by associating each element a of L with the transformation a^* carrying a general element x of L into the intersection $x \frown a$ of x with a. If the original lattice L possesses a unit element, it is easy to verify that each element a^* of L^* is identified with the element a of L into which the unit element is transformed by a^*. However, if L possesses no unit element, there will be elements of L^* that are not identified with elements of L.

The connectivity theory of the lattice L may now be developed as follows. A *covering pair* (a, c) of L is any ordered pair of elements a and c of L^* such that their union is the unit element $a \smile c = 1$. The *barrier* b of (a, c) is the intersection of the two members of the pair $b = a \frown c$. The *chains* of the lattice L are polynomials in the marks a, b, c associated

with the various covering pairs, with integer coefficients. We define the *sum* and *product* of two chains the *boundary* of a chain and the *locus* of a chain in the manner described in a recent paper by the author.[1] The only difference is that the locus of a chain is now an element of the lattice L^*. There are two main types of chains: (*a*) chains with arbitrary loci, (*b*) chains such that their loci are identified with elements of L. Chains of type (*a*) correspond to the closed chains in the paper referred to in the footnote, chains of type (*b*) to the bicompact chains. However, the present theory is much more general than the old since it does not presuppose the idea of bicompactness.

In a recent note to these PROCEEDINGS, the author gave a generalization of the notion of a topological space in which a distinction was made between the closed and the closed, bounded sets of a space.[2] The elements of L^* play the rôle of the closed sets, the elements of L of the closed, bounded sets.

The terminology used in the papers referred to in the two footnotes was not always happily chosen and will ultimately be revised. In particular, there is, of course, no connection whatever between a bounded chain (i.e., a chain with a chain boundary different from zero) and a chain having as locus a bounded set.

[1] "A Theory of Connectivity in Terms of Gratings," *Ann. Math.*, **39**, 883–912 (1938).
[2] "On the Concept of a Topological Space," these PROCEEDINGS, **25**, 52–54 (1939).

CHARACTERIZATION OF THE MOEBIUS GROUP OF CIRCULAR TRANSFORMATIONS

BY EDWARD KASNER AND JOHN DE CICCO

DEPARTMENTS OF MATHEMATICS, COLUMBIA UNIVERSITY AND BROOKLYN COLLEGE

Communicated March 1, 1939

I. *Introduction.*—The Moebius group of circular transformations is ordinarily defined as the group of real point transformations of the real cartesian plane for which the *entire* family of ∞^3 circles, or the equivalent differential equation $y''' = 3y'y''^2/(1 + y'^2)$, is invariant. The question suggests itself whether this definition does not contain redundancy—whether it is not sufficient to require that only *some* circles shall become circles.[1] If we understand by a *simple family* (∞^2) of circles, any doubly infinite family possessing the property that there is one and only one circle of the family containing a given lineal element of the plane, then our main result may be stated: *If three simple families* ($3\infty^2$) *of circles become circles under a point transformation, then the same is true for all circles, and the*

point transformation is, therefore, a circular transformation. We shall also find that the group of all point transformations in the plane may be classified into *three* distinct types with respect to the maximum number of circles which become circles. The final results are stated in Theorems 1, 2 and 3. *If a real conformal transformation converts* $3\infty^1$ *circles into circles, it must belong to the Moebius group.*

II. *Circles into Circles.*—Let T be the real point transformation

$$X = \phi(x, y), \; Y = \psi(x, y)$$

with a non-vanishing jacobian. We wish to find out what circles under T become circles. For these circles, we know that $d\kappa/ds = 0$ must be carried into $dK/dS = 0$ where κ and s are the curvature and arc length of the circle and K and S are the curvature and arc length of the transformed circle. Upon applying these conditions to the point transformation T, we find that the circles which under T become circles must satisfy the differential equation of the second order[2]

$$\bar{A} + (1 + p^2)^{3/2}\bar{B}\kappa + (1 + p^2)^2[\bar{C}(1 + p^2) + 3p\bar{E}]\kappa^2 = 0, \quad (1)$$

where κ is the curvature and $p = y'$, and where $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{E}$ are given by

$$\begin{aligned} \bar{A} &= (1 + \chi^2)A - 3\chi(\chi_x + p\chi_y)^2(\phi_x + p\phi_y), \\ \bar{B} &= (1 + \chi^2)B - 6\chi\chi_p(\chi_x + p\chi_y)(\phi_x + p\phi_y), \\ \bar{C} &= (1 + \chi^2)C - 3\chi\chi_p{}^2(\phi_x + p\phi_y), \\ \bar{E} &= (1 + \chi^2)E, \end{aligned} \quad (2)$$

and where A, B, C, E, χ are given by

$$\begin{aligned} A &= (\phi_x + p\phi_y)(\chi_{xx} + 2p\chi_{xy} + p^2\chi_{yy}) - (\chi_x + p\chi_y)(\phi_{xx} + 2p\phi_{xy} + p^2\phi_{yy}), \\ B &= (\phi_x + p\phi_y)(\chi_y + 2\chi_{xp} + 2p\chi_{yp}) - (\chi_x + p\chi_y)\phi_y - \chi_p(\phi_{xx} + 2p\phi_{xy} + p^2\phi_{yy}), \\ C &= (\phi_x + p\phi_y)\chi_{pp} - \chi_p\phi_y, \\ E &= \chi_p(\phi_x + p\phi_y), \\ \chi &= \frac{\psi_x + p\psi_y}{\phi_x + p\phi_y}. \end{aligned} \quad (3)$$

Let us find the conditions on the point transformation T which make the coefficient of κ^2 in (1) zero. Evidently for this coefficient to be zero, we must have

$$(1 + p^2)\bar{C} + 3p\bar{E} = 0. \quad (4)$$

Upon substituting (2) and (3) into (4), we find that (4) becomes

$$\begin{aligned} &\phi_x(\phi_x\phi_y + \psi_x\psi_y) + (2\phi_x\phi_y^2 + \phi_y\psi_x\psi_y - \phi_x^3 - \phi_x\psi_x^2 + \phi_x\psi_y^2)p + \\ &(\phi_y\psi_y^2 + \phi_y^3 - 2\phi_x^2\phi_y - \phi_x\psi_x\psi_y - \phi_y\psi_x^2)p^2 - \phi_y(\phi_x\phi_y + \\ &\psi_x\psi_y)p^3 = 0. \end{aligned} \quad (5)$$

Since (5) is an identity in p, we thus obtain the four conditions

$$\begin{aligned}
&\phi_x(\phi_x\phi_y + \psi_x\psi_y) = 0,\\
&2\phi_x\phi_y^2 + \phi_y\psi_x\psi_y - \phi_x^3 - \phi_x\psi_x^2 + \phi_x\psi_y^2 = 0,\\
&\phi_y\psi_y^2 + \phi_y^3 - 2\phi_x^2\phi_y - \phi_x\psi_x\psi_y - \phi_y\psi_x^2 = 0,\\
&\phi_y(\phi_x\phi_y + \psi_x\psi_y) = 0.
\end{aligned} \tag{6}$$

We find that the only functions ϕ and ψ which satisfy the above four equations are those which satisfy the equations

$$\phi_x = \pm\psi_y, \quad \phi_y = \mp\psi_x. \tag{7}$$

That is, for the coefficient of κ^2 of equation (1) to be zero, it is necessary and sufficient that the transformation T be conformal (direct or reverse). We may now state

THEOREM 1. *Let the point transformation* T *be not conformal (direct or reverse). Then under* T, *there are at most either (a) two simple families* $(2\infty^2)$ *of circles, or (b) one simple family* (∞^2) *of circles, or (c) no circles, which become circles.*

Now let T be conformal (direct or reverse). To find out what circles under T become circles, we could substitute (7) into (1), (2) and (3). But it is simpler to start over again and derive the equation (1) under the assumption that T is a conformal transformation. In the first place, let m be defined by the equation

$$m^2 = \frac{1}{J} = \frac{1}{\phi_x^2 + \phi_y^2}, \tag{8}$$

where J is the jacobian of the conformal transformation T. We then find that the circles which become circles under the conformal transformation T (direct or reverse) must satisfy the differential equation of first order[3]

$$2m_{xy}\cos 2\theta + (m_{yy} - m_{xx})\sin 2\theta = 0, \tag{9}$$

where $\tan\theta = p = y'$. By this equation, we observe that if the coefficient of κ^2 in equation (1) is identically zero, then the coefficient of κ must also be identically zero. From the equation (9), we find

THEOREM 2. *Let* T *be a conformal transformation but not a circular transformation. Then there are at most two one-parameter families* $(2\infty^1)$ *circles which become circles. Moreover the two circles of these two families through any point are orthogonal.*

(Of course, in the complex cartesian plane, the conformal transformation T of Theorem 2 converts exactly two other two one-parameter families of circles into circles. These are the minimal lines $x \pm iy =$ constant which become minimal lines. Thus a complex conformal transformation has at most $4\infty^1$ circles which become circles.)

Next let the conformal transformation T be such that the coefficients of $\cos 2\theta$ and $\sin 2\theta$ in (9) are both zero. Then we must have

$$m_{xy} = 0, \qquad m_{yy} = m_{xx}. \tag{10}$$

From (10), we find that m is given by

$$m = a(x^2 + y^2) + bx + cy + d, \tag{11}$$

where a, b, c, d are real constants. Upon substituting (11) into (8), we find

$$\phi_x^2 + \phi_y^2 = R^2, \tag{12}$$

where R is given by the equation

$$\frac{1}{R} = m = a(x^2 + y^2) + bx + cy + d. \tag{13}$$

From (12) there exists a function λ such that

$$\phi_x = R \cos \lambda, \qquad \phi_y = R \sin \lambda. \tag{14}$$

Since the equations (14) must be compatible and since ϕ is a harmonic function (that is, ϕ satisfies the equation $\phi_{xx} + \phi_{yy} = 0$), we find that λ must satisfy the two equations

$$\begin{aligned} \lambda_x &= \frac{R_y}{R} = -\frac{2ay + c}{a(x^2 + y^2) + bx + cy + d}, \\ \lambda_y &= -\frac{R_x}{R} = \frac{2ax + b}{a(x^2 + y^2) + bx + cy + d}. \end{aligned} \tag{15}$$

Since the above equations must be compatible, we find that

$$b^2 + c^2 - 4ad = 0. \tag{16}$$

First let us suppose that $a = 0$. Then $b = c = 0$ and $d \neq 0$. From (15), we find that λ is constant. By (14), we see that ϕ_x and ϕ_y are both constants. Since the transformation is conformal, we find that the transformation T must be a similitude (direct or reverse).

Next let us suppose that $a \neq 0$. Let us replace a by $1/\rho$ and ρb and ρc by -2α and -2β, respectively. Then (15) becomes

$$\lambda_x = -\frac{2(y - \beta)}{(x - \alpha)^2 + (y - \beta)^2}, \qquad \lambda_y = \frac{2(x - \alpha)}{(x - \alpha)^2 + (y - \beta)^2}. \tag{17}$$

From these equations, we obtain

$$\lambda = -2 \arctan \frac{x - \alpha}{y - \beta} + \gamma, \tag{18}$$

where γ is a constant. Upon substituting (18) into (14), and taking account of (13), we find that (14) becomes

$$\phi_x = \rho \frac{\{[(y-\beta)^2 - (x-\alpha)^2]\cos\gamma + 2(x-\alpha)(y-\beta)\sin\gamma\}}{[(x-\alpha)^2 + (y-\beta)^2]^2},$$
$$\phi_y = \rho \frac{\{[(y-\beta)^2 - (x-\alpha)^2]\sin\gamma - 2(x-\alpha)(y-\beta)\cos\gamma\}}{[(x-\alpha)^2 + (y-\beta)^2]^2}. \tag{19}$$

Solving (19) for ϕ and then solving (7) for ψ, we find that T must be of the form

$$X = \rho \frac{[(x-\alpha)\cos\gamma - (y-\beta)\sin\gamma]}{(x-\alpha)^2 + (y-\beta)^2} + h,$$
$$Y = \mp \rho \frac{[(x-\alpha)\sin\gamma + (y-\beta)\cos\gamma]}{(x-\alpha)^2 + (y-\beta)^2} + k, \tag{20}$$

where ρ, α, β, γ, h, k are constants. Thus our transformation T must be either a similitude or a transformation of form (20) generated by inversions. Hence we may state

THEOREM 3. *Let the point transformation* T *convert more than two simple families* $(2\infty^2)$ *of circles into circles. That is, let* T *convert three simple families* $(3\infty^2)$ *of circles into circles. Then* T *is a circular transformation of the Moebius group* G_6, *which is a mixed six-parameter group* G_6.

Theorems 1, 2 and 3 show that the group of all point transformations is divisible into *three* distinct types with respect to the maximum number of circles which become circles. We note in conclusion that the term circle as used in this paper includes straight line but excludes point.

[1] See the analogous simple discussion for collineations. Kasner, "The Characterization of Collineations," *Bull. Am. Math. Soc.*, **9**, 545–546 (1903). It is shown that *if four simple sets* $(4\infty^1)$ *of straight lines remain straight after a point transformation then the same is true for all straight lines, and the point transformation is, therefore, a collineation.* For any other point transformation, at most three simple families $(3\infty^1)$ of straight lines become straight lines. A refined analysis of the continuity requirements is given in the Columbia dissertation of Prenowitz (*Trans. Amer. Math. Soc.*, 1937).

[2] Kasner, "Characterization of the Conformal Group and the Equilong Group by Horn Angles," *Duke Math. Jour.*, **4**, 95–106 (1938).

[3] Comenetz, "Kasner's Invariant and Trihornometry," *Am. Math. Month.*, **45**, 82–87 (1938). Comenetz's derivation is for a direct conformal transformation. But the same procedure is valid for a reverse conformal transformation.

ON THE INTERSECTIONS OF ALGEBRAIC DIFFERENTIAL MANIFOLDS

BY J. F. RITT

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

Communicated February 21, 1939

This note on algebraic differential equations deals with a peculiar situation, in which three or more functions are uniquely determined by what may very reasonably be called two conditions.

Van der Waerden has recently published a beautiful theorem to the effect that if two irreducible algebraic manifolds in projective space of n dimensions have respectively the dimensions r and s, their intersection has no irreducible component of dimension less than $r + s - n$.[1] We shall show that the state of affairs for differential equations is more complicated. There will be produced an algebraically irreducible form in the three unknowns u, v, y whose general solution, which, of course, has two arbitrary unknowns, has precisely one solution for which $y = 0$.

Indicating differentiation by a subscript, we let

$$W = uv_1 - vu_1, \qquad A = \prod_{j=-2}^{2} (u + jv).$$

Let

$$F = A + yW^2. \tag{1}$$

F is algebraically irreducible. We shall prove that *the general solution* $\mathfrak{M}$ *of* F *has one and only one solution for which* y = 0, *namely, the solution* u = v = y = 0.

We show first that $u = v = y = 0$ is in $\mathfrak{M}$. Every solution of F which is not in $\mathfrak{M}$ annuls W. Let $\bar{u}, \bar{v}, \bar{y}$, with $W \neq 0$, be in $\mathfrak{M}$. Because F and W are homogeneous, $c\bar{u}, c\bar{v}, c\bar{y}$, where c is any constant distinct from zero, is in $\mathfrak{M}$. Making c small, we find that $u = v = y = 0$ is uniformly approximable by solutions in $\mathfrak{M}$ and hence belongs to $\mathfrak{M}$.

Let $B_j = A/(u + jv)$, $\quad j = -2, \ldots, 2$. We have

$$F = B_j(u + jv) + y[(u + jv)v_1 - (u_1 + jv_1)v]^2. \tag{2}$$

Differentiating (2) once and carrying out a fairly simple elimination, one finds that F is held by a form

$$(u_1 + jv_1)[B_j^2 + yM_j + y_1N_j]. \tag{3}$$

The form in brackets in (3) holds $\mathfrak{M}$. Thus, for a solution in $\mathfrak{M}$ with $y = 0$, one must have $B_j = 0$ for every j. Hence $u + jv$ is zero for two values of j and $u = v = 0$.

Without calculation, but with the use of special theorems, one can proceed as follows. A solution of F with $y = 0$ annuls some $u + jv$. One sees from (2) that the five manifolds of the $u + jv$ are essential for F.[2] Also, any solution of $u + jv$ which belongs to some manifold held by F but not by $u + jv$ must annul B_j. This is enough for the proof.

The principle underlying the above example permits the construction of forms in y and any number of other unknowns whose general solutions contain precisely one solution for which $y = 0$.

[1] van der Waerden, B. L., *Math. Annalen*, **115**, 330–332 (1938).
[2] Ritt, J. F., *Am. Jour. Math.*, **60**, 1–43, see p. 15 (1938).

THE PLATEAU PROBLEM FOR NON-RELATIVE MINIMA

BY MAX SHIFFMAN

DEPARTMENT OF MATHEMATICS, NEW YORK UNIVERSITY

Communicated March 6, 1939

1. The purpose of this note is to discuss minimal surfaces bounded by a given curve Γ which are not relative minima in area, i.e., are of the minimax type.[1] The note restricts itself to a special class of boundary curves Γ (described in § 3), a class which includes curves having a continuously turning tangent line. Under this restriction, we shall indicate the proofs of the following two main results: (1) if Γ bounds two minimal surfaces which are relative minima, it bounds at least one minimal surface which is not a relative minimum; (2) the Morse relations apply to the Plateau problem.

2. The procedure which we shall adopt for obtaining minimal surfaces is to consider them as extremal surfaces for the Dirichlet functional[2]

$$D[\mathfrak{x}] = 1/2 \int\int (\mathfrak{x}_u^2 + \mathfrak{x}_v^2) du dv.$$

We shall consider surfaces $\mathfrak{x} = \mathfrak{x}(r, \theta)$ defined over the unit circle $r \leq 1$, which are continuous and have piecewise continuous first derivatives, which map the boundary $r = 1$ monotonically on the given curve Γ, which map three given points θ_1, θ_2, θ_3 of $r = 1$ into three given points A, B, C of Γ, and which have a finite Dirichlet integral. We may limit ourselves to potential surfaces $\mathfrak{x}(r, \theta)$. The space of all such potential surfaces with the uniform metric $|\mathfrak{x}_1 - \mathfrak{x}_2| = \max_{r \leq 1} |\mathfrak{x}_1(r, \theta) - \mathfrak{x}_2(r, \theta)|$ will be denoted by $\mathfrak{P}$. The subspace of all surfaces $\mathfrak{x}$ for which $D[\mathfrak{x}] \leq N$ will be designated by $\mathfrak{P}_N$. It is easily shown that the set $\mathfrak{P}_N$ is compact and closed.[3]

3. Let $\mathfrak{x}(\theta)$ be a proper representation of Γ. We shall require the follow-

ing two conditions to be satisfied: (1) $\mathfrak{x}(\theta)$ is of bounded variation; (2) there is a τ such that $d\mathfrak{x}(\theta) \cdot d\mathfrak{x}(\phi) \geq 0$ for all θ, ϕ for which $|\theta - \phi| < \tau$. It is easily seen that if (1), (2) hold for one representation of Γ, they hold for all representations of Γ; and that there is a uniform τ, denoted by τ_N, for all surfaces in $\mathfrak{P}_N$.

4. Our first major result is the following:

MAIN THEOREM I. *Let Γ be a Jordan curve satisfying* (1), (2) *of* § 3. *If Γ bounds two minimal surfaces which are proper relative minima, it bounds at least one minimal surface which is not a proper relative minimum.*

The proof of the theorem is based on a typical maximum-minimum problem. Let $\mathfrak{x}'$, $\mathfrak{x}''$ be the two minimal surfaces which are proper relative minima. The problem is to find, among all closed connected sets C in $\mathfrak{P}$ containing both $\mathfrak{x}'$ and $\mathfrak{x}''$, one for which the least upper bound of $D[\mathfrak{x}]$ for all $\mathfrak{x}$ on C is the smallest possible, d.

It will be shown in § 6 that there is an admissible set C on which $D[\mathfrak{x}]$ has a finite least upper bound. One can then easily show, using the lower semi-continuity of $D[\mathfrak{x}]$, that a minimizing set C_m exists. There remains for the proof of main theorem I to establish the existence of the required minimal surface on C_m.

5. We shall use in an essential way a certain expression for the Dirichlet functional of a potential surface in terms of its boundary values. Let $\mathfrak{x}(r, \theta)$ be a potential surface defined over the unit circle with boundary values $\mathfrak{x}(\theta)$. If $\mathfrak{x}(\theta)$ is continuous and of bounded variation, we have $a_n = \frac{1}{\pi} \int \mathfrak{x}(\theta) \cos n\theta \, d\theta = -\frac{1}{\pi n} \int \sin n\theta \, d\mathfrak{x}(\theta)$, similarly for b_n, where a_n, b_n are the Fourier coefficients of $\mathfrak{x}(\theta)$. Substitute in the well-known expression $D_{r \leq \rho}[\mathfrak{x}] = \frac{\pi}{2} \Sigma n(a_n^2 + b_n^2)\rho^{2n}$; let $\rho \to 1$, using the assumptions made in § 3. One then obtains

$$D[\mathfrak{x}] = \frac{1}{4\pi} \int\int \log \frac{1}{\sin^2 1/2\,(\theta - \phi)} \, d\mathfrak{x}(\theta) d\mathfrak{x}(\phi).$$

6. Let $\mathfrak{x}(r, \theta)$, $\mathfrak{y}(r, \theta)$ be two surfaces in $\mathfrak{P}$ with boundary values $\mathfrak{x}(\theta)$, $\mathfrak{y}(\theta)$, respectively. Then $\mathfrak{y}(\Theta) = \mathfrak{x}(\theta)$ where $\Theta = \lambda(\theta)$ and $\lambda(\theta)$ is a monotonic function of θ. We shall join $\mathfrak{x}(r, \theta)$ and $\mathfrak{y}(r, \theta)$ by a 'linear' family of surfaces all belonging to $\mathfrak{P}$. Define $\mathfrak{y}_\epsilon(r, \theta)$ for ϵ in the interval $0 \leq \epsilon \leq 1$ as the potential surface with boundary values $\mathfrak{y}_\epsilon(\theta)$ given by

$$\mathfrak{y}_\epsilon(\Theta) = \mathfrak{x}(\theta) \text{ for } \Theta = (1 - \epsilon)\theta + \epsilon\lambda(\theta).$$

In the integral for $D[\mathfrak{y}_\epsilon]$, which contains $d\mathfrak{y}_\epsilon(\Theta) d\mathfrak{y}_\epsilon(\Phi)$, set $\Theta = (1 - \epsilon)\theta + \epsilon\lambda(\theta)$, $\Phi = (1 - \epsilon)\phi + \epsilon\lambda(\phi)$. This yields

$$D[\mathfrak{y}_\epsilon] = \frac{1}{4\pi} \iint \log \frac{1}{\sin^2 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta)-\lambda(\phi))]} d\mathfrak{x}(\theta) d\mathfrak{x}(\phi).$$

A similar expression can be obtained involving $d\mathfrak{y}(\theta)d\mathfrak{y}(\phi)$.

The following three lemmas are easily proved.

LEMMA 1. $D[\mathfrak{y}_\epsilon]$ *is finite and is a continuous function of* ϵ *in the closed interval* $0 \leq \epsilon \leq 1$.

LEMMA 2. $D[\mathfrak{y}_\epsilon] \leq N + \frac{L^2}{2\pi} \log \frac{4}{\sin^2 1/2\tau_N}$, *where* N *is the larger of the two quantities* $D[\mathfrak{x}]$ *and* $D[\mathfrak{y}]$, τ_N *is the quantity defined in* § 3 *and* L *is the length of the curve* Γ.

LEMMA 3. $D[\mathfrak{y}_\epsilon]$ *has a continuous derivative with respect to* ϵ *in the open interval* $0 < \epsilon < 1$, *and*

$$\frac{dD[\mathfrak{y}_\epsilon]}{d\epsilon} = -\frac{1}{4\pi} \iint \frac{\lambda(\theta) - \theta - (\lambda(\phi) - \phi)}{\tan 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta)-\lambda(\phi))]} d\mathfrak{x}(\theta) d\mathfrak{x}(\phi).$$

Thus, $\mathfrak{x}(r, \theta)$ and $\mathfrak{y}(r, \theta)$ can be connected by a path contained entirely in $\mathfrak{P}_N$ for sufficiently large N. This important result was required in § 4.

MAIN LEMMA 4. *Let* $\mathfrak{x}(r, \theta)$ *be any surface in* $\mathfrak{P}$ *whose boundary values are not constant on any arc,*[4] *and* $\mathfrak{y}^{(n)}(r, \theta)$, $n = 1, 2, \ldots$, *a sequence of surfaces in* $\mathfrak{P}$ *tending to* $\mathfrak{x}(r, \theta)$. *Let* $\mathfrak{y}_\epsilon^{(n)}(r, \theta)$, $0 \leq \epsilon \leq 1$, *be the linear path joining* $\mathfrak{x}$ *and* $\mathfrak{y}^{(n)}$ *as in lemmas* 1, 2, 3. *Then* $\left[\frac{dD[\mathfrak{y}_\epsilon^{(n)}]}{d\epsilon}\right]_{\epsilon=\epsilon^{(n)}} \to 0$ *as* $n \to \infty$ *implies that* $D[\mathfrak{y}_{\epsilon^{(n)}}^{(n)}] \to D[\mathfrak{x}]$.

Proof. Let $\mathfrak{x}(\theta)$, $\mathfrak{y}^{(n)}(\theta)$ be the boundary values of $\mathfrak{x}(r, \theta)$, $\mathfrak{y}^{(n)}(r, \theta)$, respectively, so that $\mathfrak{y}^{(n)}(\Theta) = \mathfrak{x}(\theta)$ for $\Theta = \lambda^{(n)}(\theta)$. The condition $\mathfrak{y}^{(n)}(r, \theta) \to \mathfrak{x}(r, \theta)$ implies that $\lambda^{(n)}(\theta) \Longrightarrow \theta$. In the following calculations, all the integrals involved are of the type $\frac{1}{4\pi} \iint \ldots d\mathfrak{x}(\theta) d\mathfrak{x}(\phi)$; this will be abbreviated by $\int$. The domain $|\theta - \phi| \geq \tau$ will be indicated by B', $|\theta - \phi| < \tau$ by B''. That part of an integral J taken over B' will be denoted by J', over B'' by J''. The superscript n is omitted.

We have

$$D[\mathfrak{y}_\epsilon] - D[\mathfrak{x}] = \int \log \frac{\sin^2 1/2(\theta-\phi)}{\sin^2 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta)-\lambda(\phi))]} = J_1' + J_1'',$$

and

$$J_1'' = \int_{B''} \left\{ \log \frac{\sin^2 1/2(\theta - \phi)}{(\theta - \phi)^2} + \log \frac{1}{\cos^2 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\theta))]} + \log \frac{(\theta-\phi)^2}{\tan^2 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\phi))]} \right\} = J_2'' + J_3'' + J_4''.$$

The only singular integral is J_4''. Using $\log x < x$, this is estimated by

$$J_4'' < 2 \int_{B''} \frac{\theta - \phi}{\tan 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\phi))]} = K''.$$

Now, $\frac{dD[\mathfrak{y}_\epsilon]}{d\epsilon} = J_5' + J_5''$, where the integrand is given in Lemma 3, and

$$J_5'' = -1/\epsilon \int_{B''} \frac{(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\phi))}{\tan 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\phi))]} + + 1/\epsilon \int_{B''} \frac{\theta - \phi}{\tan 1/2[(1-\epsilon)(\theta-\phi) + \epsilon(\lambda(\theta) - \lambda(\phi))]} = 1/\epsilon\, J_6'' + 1/2\epsilon\, K''. \quad \text{Or } K'' = 2\epsilon \frac{dD[\mathfrak{y}_\epsilon]}{d\epsilon} - 2\epsilon J_5' - 2J_6''.$$

Reinsert the superscript n and let $n \to \infty$, so that $\lambda^{(n)}(\theta)$ converges uniformly to θ. We have: $J_5' \to 0$, $J_6'' \to -\int_{B''} \frac{\theta - \phi}{\tan 1/2(\theta - \phi)} = -I''$, so that $K'' \to 2I''$; further, $J_1' \to 0$, $J_3'' \to \int_{B''} \log \frac{1}{\cos^2 1/2(\theta - \phi)} = L''$, and $\overline{\lim}\, J_4'' \leqq \lim K'' = 2I''$. Combining all these, we obtain $\overline{\lim}\, (D[\mathfrak{y}_{\epsilon^{(n)}}^{(n)}] - D[\mathfrak{x}]) \leq J_2'' + L'' + 2I''$. This inequality being true for all sufficiently small τ, let $\tau \to 0$. Then $\overline{\lim}\, (D[\mathfrak{y}_{\epsilon^{(n)}}^{(n)}] - D[\mathfrak{x}]) \leqq 0$. But $\underline{\lim}\, (D[\mathfrak{y}_{\epsilon^{(n)}}^{(n)}] - D[\mathfrak{x}]) \geq 0$ because of the lower semi-continuity of the Dirichlet functional. Hence, $\lim (D[\mathfrak{y}_{\epsilon^{(n)}}^{(n)}] - D[\mathfrak{x}]) = 0$, and Lemma 4 is proved.

Lemma 4 stated in ϵ, δ form yields

LEMMA 5. *Let $\mathfrak{x}(r, \theta)$ be any surface in $\mathfrak{P}$ whose boundary values are not constant on any arc.*[4] *For any η, there is an α and a δ such that, if $\mathfrak{y}(r, \theta)$ is any surface in the δ-neighborhood of $\mathfrak{x}$, then $D[\mathfrak{y}_\epsilon] \geq D[\mathfrak{x}] + \eta$ implies* $\left| \frac{dD[\mathfrak{y}_\epsilon]}{d\epsilon} \right| \geq \alpha$.

7. A deformation which is fundamental for the proof can now be constructed on the basis of Lemma 5. Define $f(\mathfrak{y}, t) \equiv \mathfrak{y}(r, \theta)$ if $|\mathfrak{y} - \mathfrak{x}| \geq \delta$;

if $|\mathfrak{y} - \mathfrak{x}| = \delta - \sigma\delta$, set $f(\mathfrak{y}, t) = \mathfrak{y}_{1-\sigma t}(r, \theta)$. By Lemma 5 this is seen to satisfy the conditions stated in the fundamental

THEOREM 1. *Let* $\mathfrak{x}(r, \theta)$, η, α, δ *be as in Lemma* 5. *There is a deformation* $f(\mathfrak{y}, t)$ *of the space* $\mathfrak{P}$ *in itself which has the following properties.* 1) $f(\mathfrak{y},0) = \mathfrak{y}$. 2) $f(\mathfrak{x}, t) \equiv \mathfrak{x}$; $f(\mathfrak{y}, t) \equiv \mathfrak{y}$ *if* $|\mathfrak{y} - \mathfrak{x}| \geq \delta$. 3) *Let* $|\mathfrak{y} - \mathfrak{x}| \leq \delta$. *If* $D[\mathfrak{y}] \leq D[\mathfrak{x}] + \eta$, *then* $D[f(\mathfrak{y}, t)] \leq D[\mathfrak{x}] + \eta$; *if* $D[\mathfrak{y}] > D[\mathfrak{x}] + \eta$, *then* $D[f(\mathfrak{y}, t)]$ *is a decreasing function of* t *for some interval* $0 \leq t \leq \bar{t}$ *and thereafter,* $\bar{t} \leq t \leq 1$, $D[f(\mathfrak{y}, t)] \leq D[\mathfrak{x}] + \eta$. 4) *Let* $|\mathfrak{y} - \mathfrak{x}| = \delta - \sigma\delta$, $0 \leq \sigma \leq 1$. *Then either* $D[f(\mathfrak{y}, 1)] \leq D[\mathfrak{x}] + \eta$ *or* $D[f(\mathfrak{y}, 1)] \leq D[\mathfrak{y}] - \sigma\alpha$.

8. Let $\mathfrak{x}(r, \theta)$ be a surface in $\mathfrak{P}$ not a minimal surface. By performing a variation of all surfaces $\mathfrak{y}(r, \theta)$ sufficiently near $\mathfrak{x}(r, \theta)$, similar to that of Courant, loc. cit. note 3, p. 693, 694, the following variational condition is derived.

THEOREM 2. *Let* $\mathfrak{x}(r, \theta)$ *be a surface in* $\mathfrak{P}$ *not a minimal surface. For any* $N > D[\mathfrak{x}]$, *there are two positive constants* δ *and* β *and a deformation* $f(\mathfrak{y}, t)$ *of* $\mathfrak{P}_N$ *in itself with the following properties.* 1) $f(\mathfrak{y}, 0) = \mathfrak{y}$. 2) $f(\mathfrak{y}, t) \equiv \mathfrak{y}$ *if* $|\mathfrak{y} - \mathfrak{x}| \geq \delta$. 3) *If* $|\mathfrak{y} - \mathfrak{x}| = \delta - \sigma\delta$, $0 \leq \sigma \leq 1$, *then* $D[f(\mathfrak{y}, t)] \leq D[\mathfrak{y}] - \beta\sigma t$; *in particular,* $D[f(\mathfrak{y}, 1)] \leq D[\mathfrak{y}] - \beta\sigma$.

9. One can now prove, on the basis of theorems 1, 2 that on the minimizing C_m there is a minimal surface $\mathfrak{x}$ for which $D[\mathfrak{x}] = d$. If not, every surface $\mathfrak{x}$ on C_m would be either a minimal surface for which $D[\mathfrak{x}] < d$ or not a minimal surface at all. If $\mathfrak{x}$ is a minimal surface with $D[\mathfrak{x}] < d$, construct the open neighborhood and deformation of theorem 1 for $\eta = \frac{d - D[\mathfrak{x}]}{2}$; if $\mathfrak{x}$ is not a minimal surface, construct the open neighborhood and deformation of theorem 2, where N is any fixed number $> d$. By the Heine-Borel-Lebesgue theorem, a finite number of these neighborhoods suffice to cover C_m completely. By applying the corresponding deformations successively, it is easily shown that C_m is deformed into an admissible set C_m' on which the least upper bound of $D[\mathfrak{x}]$ is less than d. But this contradicts the minimum character of C_m, and the proof of the main theorem 1 is complete.

10. Theorems 1, 2 also serve to establish the Morse relations.[5] For it follows, exactly as in § 9, that on each k-cap with cap limit a (in the terminology of Morse) there is a minimal surface $\mathfrak{x}$ for which $D[\mathfrak{x}] = a$. It remains to compute the connectivity numbers of $\mathfrak{P}$; using Lemma 2, one easily proves

THEOREM 3. *Let* $R_0, R_1, \ldots, R_n, \ldots$ *be the connectivity numbers of* $\mathfrak{P}$. *Then* $R_0 = 1, R_1 = R_2 = \ldots = R_n = \ldots = 0$.

We have therefore established, on the basis of the Morse theory, the

MAIN THEOREM II. *Let* μ_k *be the sum of the* kth *type nnmbers of all blocs of minimal surfaces*[6] *bounded by* Γ (*with the restrictions on* Γ *stated in* § 3).

Under the assumption that each μ_k, $k = 0, 1, \ldots$, *is finite, the following Morse relations hold:*

$$\begin{aligned} \mu_0 &\geq 1 \\ \mu_1 - \mu_0 &\geq -1 \\ &\vdots \\ \mu_n - \mu_{n-1} + \ldots + (-1)^n \mu_0 &\geq (-1)^n \\ &\vdots \end{aligned}$$

11. An example for the application of the theory is a Jordan curve which bounds a small strip around a pair of conjugate cuts on a torus. This curve, as pointed out by Courant, bounds at least two minimal surfaces which are proper relative minima. An example with two boundaries, which has not been taken up in this note, is the classical case of two circles with line of centers perpendicular to the planes of the circles.

[1] This note was read at the Sept., 1938, meeting of the American Mathematical Society, New York City.

[2] This is an essential step in the methods of Douglas, Radó, Courant.

[3] This follows from the lower semi-continuity of the Dirichlet functional and the equicontinuity of the boundary values of all surfaces in $\mathfrak{P}_N$. Cf. Courant, "Plateau's Problem and Dirichlet's Principle," *Ann. Math.*, **38**, 679–724 (1937), esp. pp. 690–692.

[4] The Lemma remains true without this restriction.

[5] Morse, "Analysis in the Large," notes of the Institute for Adv. Study, 1936–1937; "Functional Topology and Abstract Variational Theory," *Ann. Math.*, **38**, 386–448 (1937); and "The Calculus of Variations in the Large," *Amer. Math. Soc. Coll. Publ.*, **18**.

[6] By a bloc of minimal surfaces is meant a maximal connected set of minimal surfaces $\mathfrak{x}$ bounded by Γ for which $D[\mathfrak{x}] =$ constant. It is very unlikely that a bloc consists of more than one surface.

CONSISTENCY-PROOF FOR THE GENERALIZED CONTINUUM-HYPOTHESIS[1]

By Kurt Gödel

The Institute for Advanced Study

Communicated February 14, 1939

If M is an arbitrary domain of things in which a binary relation ϵ is defined, call "*propositional function over* M" any expression φ containing (besides brackets) only the following symbols: 1. Variables $x, y \ldots$ whose range is M. 2. Symbols $a_1 \ldots a_n$ denoting[2] individual elements of M (referred to in the sequel as "*the constants of* φ"). 3. ϵ. 4. $\sim$ (not), $\vee$ (or). 5. Quantifiers for the above variables $x, y \ldots$* Denote by M' the set of all subsets of M defined by prop. funct. $\varphi(x)$ over M. Call a function f with s variables a "*function in* M" if for any elements $x_1 \ldots x_s$ of M

$f(x_1 \dots x_s)$ is defined and is an element of M. If $\varphi(x)$ is a prop. funct. over M with the following normal form:

$$(x_1 \dots x_n)\ (\exists y_1 \dots y_m)\ (z_1 \dots z_k)\ (\exists u_1 \dots u_e)\ \dots .$$
$$L(xx_1 \dots x_n y_1 \dots y_m z_1 \dots z_k u_1 \dots u_e \dots .)$$

(L containing no more quantifiers) and if $a \in M$, then call "*Skolem-functions for φ and a*" any functions $f_1 \dots f_m\ g_1 \dots g_e \dots$ in M with resp. $n \dots n$, $n+k \dots n+k \dots$ variables such that for any elements $x_1 \dots x_n z_1 \dots z_k \dots$ of M the following is true:

$$L(ax_1 \dots x_n f_1(x_1 \dots x_n) \dots f_m(x_1 \dots x_n)\ z_1 \dots z_k$$
$$g_1(x_1 \dots x_n z_1 \dots z_k) \dots g_e(x_1 \dots x_n z_1 \dots z_k) \dots .)$$

The proposition $\varphi(a)$ is then equivalent with the existence of Skolem-fnct. for φ and a.

Now define: $M_0 = \{\Lambda\}$, $M_{\alpha+1} = M_\alpha'$, $M_\beta = \sum_{\alpha<\beta} M_\alpha$ for limit numbers β. Call a set x "*constructible*," if there exists an ordinal α such that $x \in M_\alpha$ and "*constructible of order α*" if $x \in M_{\alpha+1} - M_\alpha$. It follows immediately that: $M_\alpha \subset M_\beta$ *and* $M_\alpha \in M_\beta$ *for* $\alpha < \beta$ *and that:*

TH. 1. *$x \in y$ implies that the order of x is smaller than the order of y for any constr. sets x, y.*

It is easy to define a well-ordering of all constr. sets and to associate with each constr. set (of an arbitrary order α) a uniquely determined prop. fnct. $\varphi_\alpha(x)$ over M_α as its "*definition*" and furthermore to associate with each pair φ_α, a (consisting of a prop. fnct. φ_α over M_α and an element a of M_α for which $\varphi_\alpha(a)$ is true) uniquely determined "*designated Skolem-fnct. for φ_α, a.*"[3]

TH. 2. *Any constr. subset m of M_{ω_α} has an order $< \omega_{\alpha+1}$ (i.e., a constr. set, all of whose elements have orders $< \omega_\alpha$ has an order $< \omega_{\alpha+1}$).*

PROOF: Define a set K of constr. sets, a set O of ordinals and a set F of Skolem-fnct. by the following postulates I–VII:

I. $M_{\omega_\alpha} \subseteq K$ and $\alpha \in K$.

II. If $x \in K$, the order of x belongs to O.

III. If $x \in K$, all constants occurring in the definition of x belong to K.

IV. If $\alpha \in O$ and $\varphi_\alpha(x)$ is a prop. fnct. over M_α all of whose constants belong to K then:

1. The subset of M_α defined by φ_α belongs to K.

2. For any $y \in K \cdot M_\alpha$ the design. Skolem-fnct. for φ_α and y or $\sim \varphi_\alpha$ and y (according as $\varphi_\alpha(y)$ or $\sim \varphi_\alpha(y)$) belong to F.

V. If $f \in F$, $x_1 \dots x_n \in K$ and $(x_1 \dots x_n)$ belongs to the domain of definition of f, then $f(x_1 \dots x_n) \in K$.

VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first[4] element of $x - y$ belongs to K.

VII. No proper subsets of K, O, F satisfy I–VI.

TH. 3. *If $x \neq y$ and $x, y \in K \cdot M_{\alpha+1}$, then there exists a $z \in K \cdot M_\alpha$ such that $z \in x - y$ or $z \in y - x$.*[13]
(follows from VI and Th. 1.)

TH. 4.[5] $\overline{\overline{K + O + F}} = \aleph_\alpha$
since $\overline{\overline{M}}_{\omega_\alpha} = \aleph_\alpha$ and $K + O + F$ is obtained from $M_{\omega_\alpha} + \{m\}$ by forming the closure with respect to the operations expressed by II–VI.

Now denote by η the order type of O and by $\bar{\alpha}$ the ordinal corresponding to α in the similar mapping of O on the set of ordinals $< \eta$. Then we have:

TH. 5. *There exists a one to one mapping x' of K on M_η such that $x \in y \equiv x' \in y'$ for $x, y \in K$ and $x' = x$ for $x \in M_{\omega_\alpha}$.*

PROOF: The mapping x' (which will carry over the elements of order α of K exactly into all constr. sets of order $\bar{\alpha}$ for any $\alpha \in O$) is defined by transfinite induction on the order, i.e., we assume that for some $\alpha \in O$ an isomorphic[6] mapping f of $K \cdot M_\alpha$ on $M_{\bar{\alpha}}$[7] has been defined and prove that it can be extended to an isomorphic mapping g of $K \cdot M_{\alpha+1}$ on $M_{\bar{\alpha}+1}$[8] in the following way: At first those prop. fnct. over M_α whose constants belong to K (hence to $K \cdot M_\alpha$) can be mapped in a one to one manner on all prop. fnct. over $M_{\bar{\alpha}}$ by associating with a prop. fnct. φ_α over M_α having the constants $a_1 \ldots a_n$ the prop. fnct. $\varphi_{\bar{\alpha}}$ over $M_{\bar{\alpha}}$ obtained from φ_α by replacing a_i by a_i^1 and the quantifiers with the range M_α by quantifiers with the range $M_{\bar{\alpha}}$. Then we have:

TH. 6. $\varphi_\alpha(x) \quad \varphi_{\bar{\alpha}}(x^1)$ *for any $x \in K \cdot M_\alpha$.*

PROOF: If $\varphi_\alpha(x)$ is true, the designs of Skolem-fnct. for φ_α and x exist, belong to F (by IV, 2) and are functions in $K \cdot M_\alpha$ (by $\overline{\text{V}}$). Hence they are carried over by the mapping f into functions in $M_{\bar{\alpha}}$ which are Skolem-functions for $\varphi_{\bar{\alpha}}$, x^1, because the mapping f is isomorphic with respect to ϵ. Hence $\varphi_\alpha(x) \supset \varphi_{\bar{\alpha}}(x^1)$.

$\sim\varphi_\alpha(x) \supset \sim\varphi_{\bar{\alpha}}(x^1)$ is proved in the same way.

Now any φ_α over M_α whose constants belong to K defines an element of $K \cdot M_{\alpha+1}$ by IV, 1 and any element b of $K \cdot M_{\alpha+1}$ can be defined by such a φ_α (if $b \in M_{\alpha+1} - M_\alpha$ this follows by III, if $b \in M_\alpha$ then "$x \in b$" is such a φ_α). Hence the above mapping of the φ_α on the $\varphi_{\bar{\alpha}}$ gives a mapping g of all elements of $K \cdot M_{\alpha+1}$ on all elements of $M_{\bar{\alpha}+1}$ with the following properties:

A. *g is singlevalued*, because if φ_α, ψ_α define the same set, we have $\varphi_\alpha(x) \equiv \psi_\alpha(x)$ for $x \in M_\alpha \cdot K$, hence $\varphi_{\bar{\alpha}}(x^1) \equiv \psi_{\bar{\alpha}}(x^1)$ by Th. 1, i.e., $\varphi_{\bar{\alpha}}$ and $\psi_{\bar{\alpha}}$ also define the same set.

B. $x \in y \equiv x^1 \in g(y)$ *for $x \in K \cdot M_\alpha$, $y \in K \cdot M_{\alpha+1}$.*
(by Th. 6)

C. *g is one to one*, because if $x, y \in K \cdot M_{\alpha+1}$, $x \neq y$ then by Th. 3 there is a $z \in (x - y) + (y - x)$, $z \in K \cdot M_\alpha$, hence $z^1 \in [g(x) - g(y)] + [g(y) - g(x)]$ by B. Hence $g(x) \neq g(y)$.

D. *g is an extension of the mapping f, i.e., $g(x) = x^1$ for $x \in K \cdot M_\alpha$.*

PROOF: For any $b \in K \cdot M_\alpha$ a corresponding φ_α which defines it is $x \in b$, hence $\varphi_{\bar{\alpha}}$ is $x \in b^1$ hence $g(b) = b^1$.

E. *g maps $K \cdot M_\alpha$ exactly on $M_{\bar{\alpha}}$ (by D)[9] and therefore, $K(M_{\alpha+1} - M_\alpha)$ on $M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ by C.*

F. *g is isomorphic for ϵ, i.e., $g(x) \in g(y) \equiv x \in y$ for any $x, y \in K \cdot M_{\alpha+1}$.*

PROOF: If $x \in K \cdot M_\alpha$, this follows from B and D, if $x \in K \cdot (M_{\alpha+1} - M_\alpha)$ then $g(x) \in M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ by E, hence both sides of the equivalence are false by Th. 1.

By D and F, g is the desired extension of f and hence the existence of an isomorphic mapping x' of K on M_η follows by complete induction. Furthermore since all ordinals $< \omega_\alpha$ belong to O (by I, II) we have $\bar{\beta} = \beta$ for $\beta < \omega_\alpha$ from which it follows easily that $x = x'$ for $x \in M_{\omega_\alpha}$. This finishes the proof of Th. 5.

Now in order to prove Th. 2 consider the set m' corresponding to m in the isomorphic mapping of K on M_η. Its order is $< \eta < \omega_{\alpha+1}$, because $m' \in M_\eta$ and $\bar{\bar{\eta}} = \overline{O} \leq \aleph_\alpha$ by Th. 4. Since $x \in m \equiv x' \in m'$ for $x \in K$, we have $x \in m \equiv x \in m'$ for $x \in M_{\omega_\alpha}$ by Th. 5. Since furthermore $m \subseteq M_{\omega_\alpha}$ it follows that $m = m' \cdot M_{\omega_\alpha}$, i.e., m is an intersection of two sets of order $< \omega_{\alpha+1}$, which implies trivially that it has an order $< \omega_{\alpha+1}$.

TH. 7. *M_{ω_ω} considered as a model for set-theory satisfies all axioms of Zermelo[10] except perhaps the axiom of choice and M_Ω (Ω being the first inaccessible number) satisfies in addition the axiom of substitution, if in both cases "definite Eigenschaft" resp. "definite Relation" is identified with "prop. fnct. over the class of all sets" (with one resp. two free variables).*

Sketch of proof for M_{ω_ω}: ax. I, II are trivial, ax. VII is satisfied by $Z = M_\omega$, ax. III–V have the form $(\exists x)(u)[u \in x \equiv \varphi(u)]$, where the φ are certain prop. fnct. over M_{ω_ω}. Hence, by def. of $M_{\alpha+1}$ there exist sets x in $M_{\omega_\omega+1}$ satisfying the axioms. But from Th. 1 and Th. 2 it follows easily, that the order of x is smaller than ω_ω for the particular φ under consideration, so that there exist sets x in the model satisfying the axioms.

For M_Ω ax. I–V and VII are proved in exactly the same way and the axiom of subst. is proved by the same method as ax. III–V. Now denote by "A" the proposition "There exist no non-constructible sets"[11] by "R" the axiom of choice and by "C" the proposition "$2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α." Then we have:

TH. 8. *A $\supset$ R and A $\supset$ C.*

A $\supset$ R follows because for the constr. sets a well-ordering can be defined and A $\supset$ C holds by Th.2, because $\overline{M}_{\omega_\alpha} = \aleph_\alpha$.

Now the notion of "constr. set" can be defined and its theory developed in the formal systems of set theory themselves. In particular Th. 2 and, therefore, Th. 8 can be proved from the axioms of set theory. Denote the notion of "constr. set" relativized for a model M of set theory (i.e., defined in terms of the ϵ-relation of the model) by $constr._M$, then we have:

TH. 9. *Any element of* M_{ω_ω} *(resp.* M_Ω*) is constr.*$_{M_{\omega_\omega}}$ *(resp. constr.*$_{M_\Omega}$*);* in other words: *A is true in the models* M_{ω_ω} *and* M_Ω.

The proof is based on the following two facts: 1. The operation M' (defined on p. 220) is absolute in the sense that the operation relativized for the Model M_{ω_ω}, applied to an $x \in M_{\omega_\omega}$ gives the same result as the original operation (similarly for M_Ω). 2. The set N_α which has as elements all the M_β (for $\beta < \alpha$) is constr.$_{M_{\omega_\omega}}$ for $\alpha < \omega_\omega$ and constr.$_{M_\Omega}$ for $\alpha < \Omega$, as is easily seen by an induction on α. From Th. 9 and the provability (from the axioms of set theory) of Th. 8 it follows:

TH. 10. *R and C are true for the models* M_{ω_ω} *and* M_Ω.

The construction of M_{ω_ω} and M_Ω and the proof for Th. 7 and Th. 9 (therefore also for Th. 10) can (after certain slight modifications)[12] be accomplished in the resp. formal systems of set theory (without the axiom of choice), so that a contradiction derived from C, R, A and the other axioms would lead to a contradiction in set theory without C, R, A.

[1] This paper gives a sketch of the consistency proof for propositions 1, 2 of *Proc. Nat. Acad. Sci.*, **24**, 556 (1938), if T is Zermelo's system of axioms for set theory (*Math. Ann.*, **65**, 261) with or without axiom of substitution and if Zermelo's notion of "Definite Eigenschaft" is identified with "propositional function over the system of all sets." Cf. the first definition of this paper.

[2] It is assumed that for any element of M a symbol denoting it can be introduced.

[3] At first with each φ_α an equivalent normal form of the above type has to be associated, which can easily be done.

[4] In the well-ordering of the constr. sets.

[5] $\overline{m}$ means "power of m."

[6] I.e., $x \in y \equiv f(x) \in f(y)$. In the following proof $f(x)$ is abbreviated by x^1.

[7] I.e., of the elements of order $< \alpha$ of K on the elements of order $< \overline{\alpha}$ of M_η.

[8] I.e., of the elements of order $\leq \alpha$ of K on the elements of order $\leq \overline{\alpha}$ of M_η.

[9] Because f maps $K \cdot M_\alpha$ on $M_{\overline{\alpha}}$ by induct. assumpt.

[10] Cf. *Math. Ann.*, **65**, 261 (1908).

[11] In order to give A an intuitive meaning, one has to understand by "sets" all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders).

[12] In particular for the system without the axiom of substitution we have to consider instead of M_{ω_ω} an isomorphic image of it (with some other relation R instead of the $\in$-relation), because M_{ω_ω} contains sets of infinite type, whose existence cannot be proved without the axiom of subst. The same device is needed for proving the consistency of prop. 3, 4 of the paper quoted in footnote 1.

[13] Th. 3, 4, 5, are lemmas for the proof of Th. 2.

* Unless explicitly stated otherwise "prop. fnct." always means "propositional function with one free variable."

INFORMATION TO CONTRIBUTORS

THE PROCEEDINGS is the official organ of the NATIONAL ACADEMY OF SCIENCES and of the NATIONAL RESEARCH COUNCIL for the publication of brief accounts of important current researches of members of the ACADEMY and of the COUNCIL and of other American investigators. The PROCEEDINGS will aim especially to secure prompt publication of original announcements of discoveries and wide circulation of the results of American research among investigators in other countries and in all branches of science.

ARTICLES should be brief. The viewpoint should be comprehensive in giving the relation of the paper to previous publications of the author or of others and in exhibiting, where practicable, the significance of the work for other branches of science. Elaborate technical details of the work and long tables of data should be avoided, but authors should be precise in making clear the new results and should give some record of the methods and data upon which they are based.

MANUSCRIPTS should be prepared with a current number of the PROCEEDINGS as a model in matters of form, and should be typewritten in duplicate with double spacing, the author retaining one copy. Illustrations should be confined to text-figures of simple character, though more elaborate illustrations may be allowed in special instances to authors willing to pay for their preparation and insertion. Particular attention should be given to arranging tabular matter in a simple and concise manner.

REFERENCES to literature, numbered consecutively, will be placed at the end of the article and short footnotes should be avoided. It is suggested that references to periodicals be furnished in some detail and in general in accordance with the standard adopted for the Subject Catalogue of the International Catalogue of Scientific Literature, viz., name of author, with initials following (ordinarily omitting title of paper), abbreviated name of Journal, with place of publication, series (if any), volume, inclusive pages, year. For example: Montgomery, T. H., *J. Morph., Boston*, **22**, 731–815 (1911); or, Wheeler, W. M., *Konigsburg, Schr. physik., Ges.*, **55**, 1–142 (1914).

PAPERS by members of the ACADEMY may be sent to Edwin Bidwell Wilson, Managing Editor, Harvard School of Public Health, 55 Shattuck St., Boston 17, Mass. Papers by non-members of the ACADEMY or COUNCIL should be submitted through some member.

PROOF will not ordinarily be sent; if an author asks for proof, it will be sent with the understanding that charges for his corrections shall be billed to him. Authors are therefore requested to make final revisions on the typewritten manuscripts. The editors cannot undertake to do more than correct obvious minor errors.

REPRINTS should be ordered at the time of submission of manuscript. They will be furnished to authors at cost, approximately as follows:

SCHEDULE OF RATES FOR REPRINTS, CARRIAGE CHARGES EXTRA

No. copies	50	100	150	200	250
4 pp.	$2.10	$2.40	$2.70	$3.00	$3.30
8 pp.	4.20	4.80	5.40	6.00	6.60
Covers	1.60	2.05	2.50	2.95	3.40

CONTENTS

Page

GENETICS.—THE OCCURRENCE OF PARTHENOGENETIC DIPLOIDS IN TETRAPLOID MAIZE . . . . . *By L. F. Randolph and Harold E. Fischer* 161

PHYSIOLOGY.—THE SYNTHESIS AND DESTRUCTION OF VITAMIN B_1 BY PHYCOMYCES *By James Bonner and Edwin R. Buckman* 164

PHYSIOLOGY.—TEMPERATURE AND THE CRITICAL INTENSITY FOR RESPONSE TO VISUAL-FLICKER. III. ON THE THEORY OF THE VISUAL RESPONSE CONTOUR, AND THE NATURE OF VISUAL DUPLEXITY . . . . . *By W. J. Crozier and E. Wolf* 171

PHYSIOLOGY.—SPECIFIC CONSTANTS FOR VISUAL EXCITATION. IV. ON THE NATURE OF GENETIC DIFFERENCES. . . . . *By W. J. Crozier and E. Wolf* 176

PSYCHOLOGY.—A NOTE ON POLARITY POTENTIALS FROM THE HUMAN EYE. . . *By Donald B. Lindsley and Walter S. Hunter* 180

BOTANY.—LEAF GROWTH FACTORS II—THE ACTIVITY OF PURE SUBSTANCES IN LEAF GROWTH. . . . . *By David M. Bonner and A. J. Haagen-Smit* 184

ZOÖLOGY.—CENTRIOLES AND CHROMOSOMES IN THE ATYPICAL SPERMATOGENESIS OF VIVIPARA . . . . . *By Arthur Wagg Pollister* 189

MATHEMATICS.—AN ERGODIC THEOREM FOR *n*-PARAMETER GROUPS . . . . *By Nelson Dunford* 195

MATHEMATICS.—ON ANALOGUES OF THE BERNOULLI AND ALLIED NUMBERS . . *By H. S. Vandiver* 197

MATHEMATICS.—GENERAL SELF-ADJOINT BOUNDARY CONDITIONS FOR CERTAIN PARTIAL DIFFERENTIAL OPERATORS . . . . . *By J. W. Calkin* 201

MATHEMATICS.—ON THE GAUSSIAN LAW OF ERRORS IN THE THEORY OF ADDITIVE FUNCTIONS. . . . . *By P. Erdös and M. Kac* 206

MATHEMATICS.—ON THE CONNECTIVITY RING OF A LATTICE. . . . . *By J. W. Alexander* 208

MATHEMATICS.—CHARACTERIZATION OF THE MOEBIUS GROUP OF CIRCULAR TRANSFORMATIONS . . . . . *By Edward Kasner and John De Cicco* 209

MATHEMATICS.—ON THE INTERSECTIONS OF ALGEBRAIC DIFFERENTIAL MANIFOLDS . . . . . *By J. F. Ritt* 214

MATHEMATICS.—THE PLATEAU PROBLEM FOR NON-RELATIVE MINIMA. . . . *By Max Shiffman* 215

MATHEMATICS.—CONSISTENCY-PROOF FOR THE GENERALIZED CONTINUUM-HYPOTHESIS . . . . . *By Kurt Gödel* 220